The parameterization method for center manifolds

van den Berg, Jan Bouwe; Hetebrij, Wouter; Rink, Bob

published in
Journal of Differential Equations
2020

DOI (link to publisher)
10.1016/j.jde.2020.01.033

document version
Publisher's PDF, also known as Version of record

document license
Article 25fa Dutch Copyright Act

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 269 (2020) 2132–2184

Journal of Differential Equations

www.elsevier.com/locate/jde

The parameterization method for center manifolds

Jan Bouwe van den Berg [1], Wouter Hetebrij, Bob Rink *

Department of Mathematics, Vrije Universiteit Amsterdam, the Netherlands

Received 1 May 2019; revised 15 January 2020; accepted 26 January 2020
Available online 7 February 2020

Abstract

In this paper, we present a generalization of the parameterization method, introduced by Cabré, Fontich and De la Llave, to center manifolds associated to non-hyperbolic fixed points of discrete dynamical systems. As a byproduct, we find a new proof for the existence and regularity of center manifolds. However, in contrast to the classical center manifold theorem, our parameterization method will simultaneously obtain the center manifold and its conjugate center dynamical system. Furthermore, we will provide bounds on the error between approximations of the center manifold and the actual center manifold, as well as bounds for the error in the conjugate dynamical system.

1. Introduction

Cabré, Fontich and De la Llave introduced the parameterization method for invariant manifolds of dynamical systems on Banach spaces in [4–6]. The goal of the parameterization method is to find a parameterization of the (un)stable manifold associated with an equilibrium point of the dynamical system. This parameterization is defined as a conjugacy between a dynamical system on the (un)stable eigenspace and the original dynamical system on the Banach space. To find a unique conjugacy, the dynamical system on the (un)stable eigenspace is fixed (it is given as a polynomial map). Besides providing a new proof of the classical (un)stable manifold theorem,

* Corresponding author.
E-mail addresses: janbouwe@few.vu.nl (G.J.B. van den Berg), w.a.hetebrij@vu.nl (W. Hetebrij), b.w.rink@vu.nl (B. Rink).
1 Partially supported by NWO-VICI grant 639033109.

https://doi.org/10.1016/j.jde.2020.01.033

the method is useful for computational existence proofs of for example homoclinic and heteroclinic orbits, see [3,13,18]. The method has also been used for constructing (un)stable manifolds of periodic orbits, see [8]. Recent computational advances include applications to delay differential equations, see [11], and partial differential equations, see [16].

The goal of this paper is to generalize the parameterization method to center manifolds. As explained above, the proof of the classical parameterization method fixes the dynamics on the (un)stable eigenspace. Having explicit dynamics on the (un)stable eigenspace a priori ensures that the parameterization of the manifold is unique. For center manifolds, we generally cannot choose the dynamics on the center eigenspace ourselves. Thus besides solving for the conjugacy, we also need to solve for the dynamics on the center eigenspace. To ensure a unique parameterization, we fix part of the conjugacy instead of fixing the conjugate dynamics.

To state the main theorem of this paper, we introduce some notation. Let $F : X \to X$ be a C^n (for $n \geq 2$) discrete time dynamical system on a Banach space X, with non-hyperbolic fixed point 0. Denote $F(x) = Ax + g(x)$, with A the Jacobian of F at 0. We assume that A has a spectral gap around the unit circle, and write $X = X_c \oplus X_h$, where X_c is the eigenspace associated with the eigenvalues on the unit circle, and X_h the eigenspace associated with the eigenvalues away from the unit circle. We write $A = \begin{pmatrix} A_c & 0 \\ 0 & A_h \end{pmatrix}$ with respect to the splitting $X = X_c \oplus X_h$. Our goal is to find a conjugacy $K : X_c \to X$ and a dynamical system $R : X_c \to X_c$ such that the diagram

$$\begin{tikzcd}
X = X_c \oplus X_h \arrow[r, "F = A + g"] & X = X_c \oplus X_h \\
X_c \arrow[u, "K = \iota + \binom{k_c}{k_h}"] \arrow[r, "R = A_c + r"] & X_c \arrow[u, "K = \iota + \binom{k_c}{k_h}"']
\end{tikzcd}$$

commutes, where $\iota : X_c \to X_c \oplus X_h$ is the inclusion map. We impose that the conjugacy K is tangent to X_c at 0, i.e. $DK(0) = \binom{\mathrm{Id}}{0}$, which implies that the linearization of R at 0 is given by A_c. We also fix part of the conjugacy K by an explicit choice of the non-linear map $k_c : X_c \to X_c$. Our result is the following.

Theorem *(Parameterization of the center manifold). If $g : X \to X$ and $k_c : X_c \to X_c$ are C^n, sufficiently small in the C^1 norm and bounded in C^2 norm, see Section 2 for the explicit bounds, then there exist a C^n map $k_h : X_c \to X_h$ and a C^n map $r : X_c \to X_c$ such that*

$$(A + g) \circ K = K \circ (A_c + r) \qquad \text{where } K = \iota + \begin{pmatrix} k_c \\ k_h \end{pmatrix}. \tag{1.0.1}$$

Furthermore, we have explicit bounds on the C^1 norms of k_h and r in terms of the spectral gap of A and the C^1 norms of g and k_c. These bounds are made precise in Section 2.

Despite k_h and r being C^n, we only give explicit bounds on their C^1 norm. We note that as g is at least C^2, we may multiply g as usual with a cut-off function ξ such that $x \mapsto Ax + g(x)\xi(x)$ satisfies the conditions of Theorem 2.1. Then we find a local center manifold for $x \mapsto Ax + g(x)$ on the region where $\xi = 1$, as well as local conjugate dynamics on the center eigenspace.

Our theorem provides a new proof of the classical center manifold theorem, see [7,12,19]. The classical proof of the center manifold theorem is based on the variation of constants formula. Our proof is more elementary. It simultaneously finds k_h and r as fixed point of an operator on a function space, which is a contraction on an appropriately chosen subset. The freedom to choose $k_c \neq 0$ moreover allows us to obtain the dynamics on the center eigenspace directly in a desired normal form.

As we mentioned in the beginning, our work is inspired by the parameterization method for invariant (un)stable manifolds in [4–6], which we shall refer to as the classical parameterization method. The main difference between the classical and the proposed parameterization method is the treatment of the conjugate dynamics R. The classical parameterization method fixes a polynomial map $R : X_{u/s} \to X_{u/s}$ which determines the dynamical system on the (un)stable eigenspace, whereas our method produces a C^n map $R : X_c \to X_c$.

It follows from [10] that one can find Taylor approximations R_0 and K_0 of the maps R and K up to any finite order. These R_0 and K_0, will then almost satisfy the conjugacy equation, i.e. $F \circ K_0 - K_0 \circ R_0$ will be of order $\|x\|^m$. This will imply that the real solution of the conjugacy equation lies very close to R_0 and K_0. More precisely, we can find rigorous bounds for $R - R_0$ and $K - K_0$. Hence, from the Taylor approximations we can extract very detailed information of the true dynamical behavior on the center manifold.

Another generalization of the classical parameterization method concerning center manifolds can be found in [1,2]. This generalization finds certain submanifolds of the center manifold at parabolic fixed points. A fixed point is called parabolic if the linearization of the dynamical system is the identity. As in the classical parameterization method, the conjugate vector field is polynomial in this case, and the proof in [1,2] follows roughly the same analysis as the proof of the classical parameterization method. A drawback of this result is that it produces invariant manifolds which are only continuous at the origin. Furthermore, it imposes that the linearization of the dynamical system at the fixed point is the identity on the center eigenspace. Our theorem on the other hand produces invariant manifold which are everywhere C^n and does not impose that the linearization restricted to the center eigenspace is the identity.

1.0.1. Outline of the paper

Our paper consists of three parts. We start by introducing notation in the following section and by giving a more precise formulation of the main theorem in Section 2. As a corollary of the main theorem, we find an upper bound between the error of an almost solution $\mathcal{M}$ of (1.0.1) and the actual solution Λ of (1.0.1) in terms of how well $\mathcal{M}$ solves (1.0.1). Furthermore, we introduce a fixed point problem in Section 3. This fixed point problem will produce the conjugacy and conjugate dynamics of Theorem 2.1. Finally, we will prove that the fixed point problem defines a contraction in C^1.

In the second part of our paper, which consists of Sections 4 to 6, we will obtain the smoothness results of the main theorem by means of two bootstrapping arguments. The first bootstrapping argument, in Section 4, shows that once we have a C^1 conjugacy and conjugate dynamics, they are in fact both C^2. The second argument, in Section 5, shows that we can inductively increase the smoothness from C^2 to C^n. Finally, we prove in Section 6 that our center manifold is unique, and we derive the estimate on $\Lambda - \mathcal{M}$.

1.0.2. Future work

Our proposed method is defined for general discrete time dynamical systems. Although it is not proven in this paper, we are convinced that the method can be generalized to continuous time dynamical systems. Furthermore, our method can be used to parameterize different manifolds than only the center manifold, for instance the center-(un)stable or (un)stable manifold. In the latter case, our method should even work if the (un)stable manifold contains infinitely many resonances.

1.1. Notation and conventions

We use the following notation and conventions in this paper.

- For a Banach space $X = \bigoplus_{i=1}^n X_i$, we assume that the norm on X satisfies

$$\|x\| = \max_{1\le i\le n} \{\|x_i\|_i\}, \qquad \text{for } x = \oplus_{i=1}^n x_i \tag{1.1.1}$$

  where $x_i \in X_i$ and $\|\cdot\|_i$ the norm on X_i. If X is equipped with another norm, we can always define an equivalent norm on X which satisfies (1.1.1) and leaves the norm unchanged on X_i.
- For functions $f : X \to Y$ between Banach spaces, we denote with

$$\|f\|_n := \max_{0\le m\le n} \sup_{x\in X} \|D^m f(x)\|$$

  the C^n norm of f for $n \ge 0$.

  For X and Y Banach spaces, we denote with

$$C_b^n(X,Y) := \left\{ f : X \to Y \mid f \text{ is } C^n \text{ and } \|f\|_n < \infty \right\}$$

  the Banach space of all C^n bounded functions between X and Y.
- For a linear operator $A : X \to Y$ between Banach spaces, we denote with

$$\|A\|_{\mathrm{op}} := \sup_{\|x\|=1} \|Ax\|$$

  the operator norm of A.

  For X and Y Banach spaces, we denote with

$$\mathcal{L}(X,Y) := \left\{ A : X \to Y \mid A \text{ is a linear operator and } \|A\|_{\mathrm{op}} < \infty \right\}$$

  the Banach space of all bounded linear operators between X and Y.
- For a k-linear operator $A : X^k \to Y$ between Banach spaces, we denote with

$$\|A\|_{\mathrm{op}} := \sup_{\substack{\|x_i\|\le 1 \\ 1\le i\le k}} \|A(x_1,\dots,x_k)\|$$

  the operator norm of A.

  For X and Y Banach spaces, we denote with

$$\mathcal{L}^k(X,Y) := \left\{ A : X^k \to Y \,\middle|\, A \text{ is a } k\text{-linear operator and } \|A\|_{\mathrm{op}} < \infty \right\}$$

  the Banach space of all bounded k-linear operators between X^k and Y.

– For Banach spaces X, Y and Z and $f \in C_b^n(Y, Z)$ and $g \in C_b^n(X, Y)$, the kth derivative of $f \circ g$ is given by

$$D^k[f \circ g](x) = D^k f(g(x))(\underbrace{Dg(x), \dots, Dg(x)}_{k \text{ times } Dg(x)}) + \text{lower derivatives of } f. \tag{1.1.2}$$

We can identify the kth derivative of f at $g(x)$ with a symmetric k-linear operator between Y^k and Z. This motivates us to define the shorthand notation

$$A^{\otimes k} := \underbrace{(A, \dots, A)}_{k \text{ times } A} \in \mathcal{L}(X^k, Y^k) \qquad \text{for } A \in \mathcal{L}(X, Y).$$

Hence we may rewrite (1.1.2) as

$$D^k[f \circ g](x) = D^k f(g(x))\,(Dg(x))^{\otimes k} + \text{lower derivatives of } f,$$

where $D^k f(g(x)) \in \mathcal{L}^k(Y, Z)$.

2. A quantitative formulation of the parameterized center manifold theorem

We will now formulate the quantitative version of the parameterization of the center manifold theorem.

Theorem 2.1 *(Parameterization of the center manifold). Let X be a Banach space and $F : X \to X$ a C^n, $n \geq 2$, discrete dynamical system on X such that 0 is a fixed point of F. Denote $F = A + g$ with $A := DF(0)$ and let $k_c : X_c \to X_c$ be chosen. Assume that*

1. *There exist closed A-invariant subspaces X_c, X_u and X_s such that $X = X_c \oplus X_u \oplus X_s$. We write $A = \begin{pmatrix} A_c & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_s \end{pmatrix}$ where we define $A_c := A\big|_{X_c}$, and similarly define A_u and A_s.*
2. *The linear operators A_c and A_u are invertible.*
3. *The norm on X is such that*

$$\|A_c^{-1}\|_{\mathrm{op}}^{\tilde{n}} \|A_s\|_{\mathrm{op}} < 1 \qquad \textit{and} \qquad \|A_u^{-1}\|_{\mathrm{op}} \|A_c\|_{\mathrm{op}}^{\tilde{n}} < 1 \qquad \textit{for all } 1 \leq \tilde{n} \leq n.$$

4. *The non-linearities g and k_c satisfy*

$$g \in \left\{ h \in C_b^n(X, X) \mid h(0) = 0,\ Dh(0) = 0 \textit{ and } \|Dh\|_0 \leq L_g \right\},$$
$$k_c \in \left\{ h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0 \textit{ and } \|Dh\|_0 \leq L_c \right\},$$

for L_g and L_c small enough; see Remark 2.4 for the explicit inequalities that L_g and L_c should satisfy.

Then there exist a C^n conjugacy $K : X_c \to X$ and C^n discrete dynamical system $R = A_c + r : X_c \to X_c$ such that

$$(A+g)\circ K = K\circ (A_c + r). \tag{2.0.1}$$

Furthermore, K and R have the following properties:

A) *The dynamical system $R = A_c + r$ is globally invertible, with its inverse given by $T = A_c^{-1} + t$, where*

$$r \in \left\{h \in C_b^n(X_c, X_c) \mid h(0)=0,\ Dh(0)=0 \text{ and } \|Dh\|_0 \leq L_r\right\},$$
$$t \in \left\{h \in C_b^n(X_c, X_c) \mid h(0)=0,\ Dh(0)=0 \text{ and } \|Dh\|_0 \leq L_t\right\}.$$

The constants L_r and L_t depend on L_g and L_c. See Remark 2.4 for their definition.

B) *The conjugacy K is given by*

$$K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$$

with $\iota : X_c \to X$ the inclusion map and

$$k_u \in \left\{h \in C_b^n(X_c, X_u) \mid h(0)=0,\ Dh(0)=0 \text{ and } \|Dh\|_0 \leq L_u\right\},$$
$$k_s \in \left\{h \in C_b^n(X_c, X_s) \mid h(0)=0,\ Dh(0)=0 \text{ and } \|Dh\|_0 \leq L_s\right\}.$$

The constants L_u and L_s depend on L_g and L_c. See Remark 2.4 for their definition.

Remark 2.2. If A has a spectral gap around the unit circle, conditions 1 and 2 hold when we take X_c to be the center eigenspace, X_u to be the unstable eigenspace and X_s to be the stable eigenspace. Furthermore, in this case there exists a norm on X such that condition 3 holds. For the construction of this norm, see Proposition A.1 of [4].

In many practical situations, using for example numerics or Taylor approximations, one may be able to find K_0 and R_0 which almost solve (2.0.1), i.e. for which $F\circ K_0 - K_0\circ R_0$ is small. In other instances, the dynamical system $F : X \to X$ may be close to another dynamical system $F_0 : X \to X$, for which we are able to find a K_0 and R_0 satisfying $F_0\circ K_0 - K_0 \circ R_0 = 0$ exactly. In both cases, we are interested in estimating the error between K_0 and the true conjugacy K, as well as the error between R_0 and the true dynamical system R. The following corollary of Theorem 2.1 gives bounds for these errors.

Corollary 2.3. *Let $n \geq 2$ and let $F : X \to X$ and $k_c : X_c \to X_c$ be C^n functions which satisfy the conditions of Theorem 2.1. Let $\varepsilon > 0$, $m < n$ and $M > 0$. Then there exists a constant $\mathcal{C}(M, m)$, which will be introduced in Section 6.3, such that if*

1. $k_0 \in \left\{h \in C_b^{m+1}(X_c, X_u \oplus X_s) \;\middle|\; h(0)=0,\ Dh(0)=0 \text{ and } \|h\|_{m+1} \leq M\right\},$
2. $r_0 \in \left\{h \in C_b^{m+1}(X_c, X_c) \;\middle|\; h(0)=0,\ Dh(0)=0,\ \|Dh\|_0 \leq L_r \text{ and } \|h\|_{m+1} \leq M\right\},$
3. $\left\| F\circ\left(\iota + \begin{pmatrix} k_c \\ k_0\end{pmatrix}\right) - \left(\iota + \begin{pmatrix} k_c \\ k_0\end{pmatrix}\right)\circ (A_c + r_0)\right\|_m \leq \varepsilon,$

then we have

$$\|k-k_0\|_m \le C(M,m)\varepsilon \qquad \text{and} \qquad \|r-r_0\|_m \le C(M,m)\varepsilon$$

for $k = k_u \oplus k_s : X_c \to X_u \oplus X_s$ and $r : X_c \to X_c$ from Theorem 2.1.

Remark 2.4. To state what it means in Theorem 2.1 for L_g and L_c to be small enough, we first introduce explicit formulas for L_r, L_t, L_u and L_s:

$$L_r := \frac{L_g + L_c\left(2\|A_c\|_{\mathrm{op}} + L_g\right)}{1-L_c}, \tag{2.0.2a}$$

$$L_t := \frac{\|A_c^{-1}\|_{\mathrm{op}}^2 L_r}{1-\|A_c^{-1}\|_{\mathrm{op}} L_r}, \tag{2.0.2b}$$

$$L_u := \frac{\|A_u^{-1}\|_{\mathrm{op}}\,(1+L_c)\,L_g}{1-L_r\|A_u^{-1}\|_{\mathrm{op}} - \|A_c\|_{\mathrm{op}}\|A_u^{-1}\|_{\mathrm{op}}}, \tag{2.0.2c}$$

$$L_s := \frac{\|A_c^{-1}\|_{\mathrm{op}}\,(1+L_c)\,L_g}{1-L_r\|A_c^{-1}\|_{\mathrm{op}} - \|A_s\|_{\mathrm{op}}\|A_c^{-1}\|_{\mathrm{op}}}. \tag{2.0.2d}$$

For L_r, L_t, L_u, and L_s to be positive, their denominators have to be positive. So we will first of all require L_g and L_c to be so small that

$$\begin{aligned} L_c &< 1, &\qquad L_r\|A_u^{-1}\|_{\mathrm{op}} + \|A_c\|_{\mathrm{op}}\|A_u^{-1}\|_{\mathrm{op}} &< 1, \\ L_r\|A_c^{-1}\|_{\mathrm{op}} &< 1, &\qquad L_r\|A_c^{-1}\|_{\mathrm{op}} + \|A_s\|_{\mathrm{op}}\|A_c^{-1}\|_{\mathrm{op}} &< 1. \end{aligned}$$

Besides these conditions on L_g and L_c, the proof of Theorem 2.1 contains multiple fixed points arguments, with contraction constants depending on L_g and L_c. The corresponding contraction constants are, for $0 \le \tilde{n} \le n$,

$$\theta_{\tilde{n},1} := L_g + L_c, \tag{2.0.3a}$$

$$\theta_{\tilde{n},2} := \|A_u^{-1}\|_{\mathrm{op}}\left(\left(\|A_c\|_{\mathrm{op}} + L_r\right)^{\tilde{n}} + L_g + L_u\right), \tag{2.0.3b}$$

$$\theta_{\tilde{n},3} := L_{-1}^{\tilde{n}}\left(\|A_s\|_{\mathrm{op}}\left(1+L_{-1}L_s\right) + L_g\left(1+L_{-1}\left(1+L_c\right)\right)\right). \tag{2.0.3c}$$

We note that $\theta_{\tilde{n},1}$ does not depend on $\tilde{n}$, but for consistency we choose to keep the index $\tilde{n}$. Furthermore, we define

$$L_{-1} := \|A_c^{-1}\|_{\mathrm{op}} + L_t,$$

which will act as a bound on $\|D(A_c + r)^{-1}\|_0$. Now, if L_g and L_c are small, then we have that $\theta_{\tilde{n},i} < 1$. In particular, we require L_g and L_c to be so small that indeed

$$\theta_{\tilde{n},1} < 1, \qquad \theta_{\tilde{n},2} < 1, \qquad \theta_{\tilde{n},3} < 1 \qquad \text{for all } \tilde{n} \le n.$$

We will often refer to this remark by writing "(...) small in the sense of Remark 2.4 for $n = m$.". By this, we mean that $\theta_{\tilde{n},i} < 1$ holds for all $\tilde{n} \leq m$ and $i = 1, 2, 3$.

2.1. *Proof scheme of Theorem 2.1*

To prove Theorem 2.1 we use four steps:

1. We define a fixed point operator and show that its fixed points are solutions to the conjugacy equation (2.0.1).
2. We show that the fixed point operator is a contraction with respect to the C^1 norm on an appropriate set of C^2 functions. Therefore, we find a pair of C^1 functions (K, r) such that equation (2.0.1) holds.
3. We show that the C^1 solution (K, r) of equation (2.0.1) is in fact C^2.
4. Finally, we use induction to prove that K and r are C^n.

We find it easiest to split the proof in the four steps outlined above. We note that in step 2 we find a contraction with respect to the C^1 norm in a space of C^2 functions, which is the reason that we assume $n \geq 2$ in Theorem 2.1. Furthermore, going from a C^1 solution to a C^2 solution needs different estimates than when going from a C^m solution to a C^{m+1} solution for $m \geq 2$, see for instance Lemma 4.3 and Lemma 5.6.

3. A C^1 center manifold

3.1. *A fixed point operator*

We start by setting up a fixed point operator. For this, we consider the conjugacy equation

$$(A + g) \circ K - K \circ (A_c + r) = 0.$$

When we write $g = \begin{pmatrix} g_c \\ g_u \\ g_s \end{pmatrix}$ and $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$, then we have component-wise

$$\begin{pmatrix} A_c & 0 & 0 \\ 0 & A_u & 0 \\ 0 & 0 & A_s \end{pmatrix} \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} + \begin{pmatrix} g_c \circ K \\ g_u \circ K \\ g_s \circ K \end{pmatrix} - \begin{pmatrix} A_c + r + k_c \circ (A_c + r) \\ k_u \circ (A_c + r) \\ k_s \circ (A_c + r) \end{pmatrix} = 0.$$

In other words, we obtain the three equations

$$\begin{cases} A_c k_c + g_c \circ K - r - k_c \circ (A_c + r) = 0, \\ A_u k_u + g_u \circ K - k_u \circ (A_c + r) = 0, \\ A_s k_s + g_s \circ K - k_s \circ (A_c + r) = 0. \end{cases} \tag{3.1.1}$$

The first equation is equivalent to

$$r = A_c k_c + g_c \circ K - k_c \circ (A_c + r).$$

Secondly, as A_u is invertible, the second equation is equivalent to

$$k_u = A_u^{-1}(A_u k_u) = A_u^{-1}(k_u \circ (A_c + r) - g_u \circ K).$$

Finally, if we assume that $A_c + r$ is invertible, the last equation is equivalent to

$$k_s = A_s k_s \circ (A_c + r)^{-1} + g_s \circ K \circ (A_c + r)^{-1}.$$

We see that the system (3.1.1) is equivalent with

$$\begin{cases} r = A_c k_c + g_c \circ K - k_c \circ (A_c + r), \\ k_u = A_u^{-1} k_u \circ (A_c + r) - A_u^{-1} g_u \circ K, \\ k_s = A_s k_s \circ (A_c + r)^{-1} + g_s \circ K \circ (A_c + r)^{-1}. \end{cases} \tag{3.1.2}$$

Thus $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix}$ is a fixed point of the operator Θ, defined by

$$\Theta : \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \mapsto \begin{pmatrix} A_c k_c + g_c \circ K - k_c \circ (A_c + r) \\ A_u^{-1} k_u \circ (A_c + r) - A_u^{-1} g_u \circ K \\ A_s k_s \circ (A_c + r)^{-1} + g_s \circ K \circ (A_c + r)^{-1} \end{pmatrix}. \tag{3.1.3}$$

The following proposition summarizes this derivation.

Proposition 3.1. *Let $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix} : X_c \to X$ and $A_c + r : X_c \to X_c$. If $A_c + r$ is globally invertible, then the following are equivalent:*

i) The dynamical system F is conjugate to $A_c + r$ by K.
ii) The function $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} : X_c \to X$ satisfies the system (3.1.2).
iii) The function $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} : X_c \to X$ is a fixed point of Θ defined in (3.1.3).

Proof. This follows from the derivation above. □

3.2. Θ is well-defined

One of the conditions of Proposition 3.1 is that $A_c + r$ is globally invertible. Under a mild condition on the derivative of r, we show that $A_c + r$ is a global diffeomorphism. We will state a more general result, as we need a more general statement in Proposition 6.3. Furthermore, we note that the result also follows from theorem 2.1 of [15], which states that under some growth condition on $A_c + r$ and the inverse of its derivative, $A_c + r$ is a global diffeomorphism. However, we will provide an alternative proof, as this proof is exemplary for the structure of the proof of Theorem 2.1.

Lemma 3.2. *Let $B \in \mathcal{L}(X_c, X_c)$ be invertible.*

i) We have that $B + \psi : X_c \to X_c$ is a global diffeomorphism for all $\psi \in C_b^1(X_c, X_c)$ with $\|D\psi\|_0 < \|B^{-1}\|_{\mathrm{op}}^{-1}$.

ii) If $B + \psi : X_c \to X_c$ is a homeomorphism and $\psi \in C_b^0(X_c, X_c)$, then $(B + \psi)^{-1} = B^{-1} + \varphi$ with $\varphi \in C_b^0(X_c, X_c)$.

Proof. *i)* We will first show that $B + \psi$ is globally invertible, and use its inverse to prove that it is in fact a diffeomorphism, which is similar to how we show the smoothness result of Theorem 2.1.

For $\varphi \in C_b^0(X_c, X_c)$, we have $(B + \psi) \circ (B^{-1} + \varphi) = \mathrm{Id}$ if and only if

$$\varphi = -B^{-1}\left(\psi \circ \left(B^{-1} + \varphi\right)\right) =: \Psi(\varphi). \tag{3.2.1}$$

Our first goal is to show that Ψ is a contraction on $C_b^0(X_c, X_c)$.

As ψ is C^1 with bounded derivative, it is Lipschitz with Lipschitz constant equal to the norm of its derivative. Thus for $\varphi, \tilde{\varphi} \in C_b^0(X_c, X_c)$ we have

$$\|\Psi(\varphi) - \Psi(\tilde{\varphi})\|_0 \leq \|B^{-1}\|_{\mathrm{op}} \|D\psi\|_0 \|\varphi - \tilde{\varphi}\|_0.$$

Thus with $\|D\psi\|_0 < \|B^{-1}\|_{\mathrm{op}}^{-1}$ we have that Ψ is contraction operator. Let φ denote the fixed point of Ψ.

If φ is C^1, we can differentiate (3.2.1) to x, and the derivative of φ satisfies

$$D\varphi = -B^{-1} D\psi(B^{-1} + \varphi)(B^{-1} + D\varphi). \tag{3.2.2}$$

In other words, if φ is C^1, then $D\varphi$ is a fixed point of the operator

$$\begin{aligned} \Phi : C_b^0(X_c, \mathcal{L}(X_c, X_c)) &\to C_b^0(X_c, \mathcal{L}(X_c, X_c)), \\ A &\mapsto -B^{-1} D\psi(B^{-1} + \varphi) B^{-1} - B^{-1} D\psi(B^{-1} + \varphi) A. \end{aligned}$$

We will show that Φ is a contraction, and that its fixed point is indeed the derivative of φ.

As $\|B^{-1} D\psi(B^{-1} + \varphi)\|_0 \leq \|B\|_{\mathrm{op}} \|D\psi\|_0 < 1$, we see that Φ is contraction, and we denote the fixed point of Φ by $\mathcal{A}$. All that is left to show is that $\mathcal{A}$ is the derivative of φ. We have

$$\begin{aligned} &\psi(B^{-1}(x + y) + \varphi(x + y)) - \psi(B^{-1}x - \varphi(x)) \\ &\qquad = \int_0^1 D\psi(z(s, x, y))\mathrm{d}s(B^{-1}y + \varphi(x + y) - \varphi(y)), \end{aligned}$$

where we define $z(s, x, y) := B^{-1}x + sB^{-1}y + s\varphi(x + y) + (1 - s)\varphi(x)$. We find, as ϕ is the fixed point of Ψ,

$$\begin{aligned} &\|\varphi(x + y) - \varphi(y) - \mathcal{A}(x)y\| \\ &\qquad \leq \|B^{-1}\|_{\mathrm{op}} \int_0^1 \|D\psi(z(s, x, y))\|_{\mathrm{op}}\mathrm{d}s \|(\varphi(x + y) - \varphi(x) - \mathcal{A}(x)y)\| \end{aligned}$$

$$+ \|B^{-1}\|_{\mathrm{op}} \int_0^1 \|D\psi(z(s,x,y)) - D\psi(z(s,x,0))\|_{\mathrm{op}} \mathrm{d}s \|B^{-1}y\|$$

$$+ \|B^{-1}\|_{\mathrm{op}} \int_0^1 \|D\psi(z(s,x,y)) - D\psi(z(s,x,0))\|_{\mathrm{op}} \mathrm{d}s \|\mathcal{A}(x)y\|. \quad (3.2.3)$$

Define $\theta = \|B^{-1}\|_{\mathrm{op}} \int_0^1 \|D\psi(z(s,x,y))\|\mathrm{d}s$, then $\theta \le \|B^{-1}\|_{\mathrm{op}}\|D\psi\|_0 < 1$. Hence (3.2.3) implies

$$\frac{\|\varphi(x+y) - \varphi(y) - \mathcal{A}(x)y\|}{\|y\|}$$
$$\le \frac{\|B^{-1}\|_{\mathrm{op}}}{1-\theta} \int_0^1 \|D\psi(z(s,x,y)) - D\psi(z(s,x,0))\|_{\mathrm{op}} \mathrm{d}s (\|B^{-1}\|_{\mathrm{op}} + \|\mathcal{A}(x)\|_{\mathrm{op}}).$$

Furthermore, $\|D\psi\|_0$ is finite, hence by using the Lebesgue Dominated Convergence Theorem, we can interchange limits and integrals in

$$\lim_{\|y\|\to 0} \int_0^1 \|D\psi(z(s,x,y)) - D\psi(z(s,x,0))\|_{\mathrm{op}} \mathrm{d}s.$$

Using continuity of $\|D\psi(z(s,x,y))\|_{\mathrm{op}}$ in y, the limit is 0. Therefore,

$$\lim_{\|y\|\to 0} \frac{\|\varphi(x+y) - \varphi(y) - \mathcal{A}(x)y\|}{\|y\|} = 0$$

and we conclude that the right inverse $B^{-1} + \varphi$ of $B + \psi$ is C^1.

We still need to show that $B^{-1} + \varphi$ is the inverse of $B + \psi$, or equivalently show that $B + \psi$ is injective. Let $x, y \in X_c$ be such that $Bx + \psi(x) = By + \psi(y)$, then

$$0 = \|Bx + \psi(x) - By - \psi(y)\| \ge \|B^{-1}\|_{\mathrm{op}}^{-1}\|x - y\| - \|D\psi\|_0\|x - y\|.$$

As $\|B^{-1}\|_{\mathrm{op}}^{-1} - \|D\psi\|_0 > 0$, we must have $\|x - y\| = 0$, and thus $x = y$. With this, we have shown that $B + \psi$ is injective, and thus invertible.

ii) We want to show that for all bounded continuous $\psi : X_c \to X_c$ such that $B + \psi$ is a homeomorphism, $\varphi := B^{-1} - (B + \psi)^{-1}$ is uniformly bounded. We have

$$\begin{aligned}
\sup_{x\in X_c} \|\varphi(x)\| &= \sup_{x\in X_c} \left\| \left(B^{-1} - (B+\psi)^{-1}\right)(x)\right\| \\
&= \sup_{x\in X_c} \left\| \left(B^{-1} - (B+\psi)^{-1}\right)((B+\psi)(x))\right\| \\
&= \sup_{x\in X_c} \|x + B^{-1}\psi(x) - x\|
\end{aligned}$$

$$\begin{aligned}&\leq \sup_{x\in X_c}\|B^{-1}\|_{\mathrm{op}}\|\psi(x)\| \\ &= \|B^{-1}\|_{\mathrm{op}}\|\psi\|_0. \quad \square\end{aligned}$$

From the above lemma it follows that if $r : X_c \to X_c$ is C^1 and its derivative Dr is uniformly bounded by $\|A_c^{-1}\|_{\mathrm{op}}^{-1}$, then there exists a C^1 bounded function $t : X_c \to X_c$ such that $(A_c + r)^{-1} = A_c^{-1} + t$. Furthermore, from equation (3.2.2) it follows that $\|Dt\|_0$ satisfies

$$\|Dt\|_0 \leq \|A_c^{-1}\|_0\|Dr\|_0\|A_c^{-1}\|_0 + \|A_c^{-1}\|_0\|Dr\|_0\|Dt\|_0.$$

In particular, we claim in Theorem 2.1 that the dynamical system $A_c + r$ will be invertible with inverse $A_c^{-1} + t$. Furthermore, we claim that we have the bounds $\|Dr\|_0 \leq L_r$ and $\|Dt\|_0 \leq L_t$, with L_r and L_t defined in (2.0.2a) and (2.0.2b) respectively. Since $L_r \leq \|A_c^{-1}\|_{\mathrm{op}}^{-1}$, it follows from the lemma that if $\|Dr\|_0 \leq L_r$, we have that $A_c + r$ is invertible, and from the definition of L_t that $\|Dt\|_0 \leq L_t$. In particular, Θ is well-defined if we assume $\|Dr\|_0 \leq L_r$ and the desired properties of $R = A_c + r$ follow from the above discussion.

3.3. *A first invariant set for* Θ

We want to find an invariant set for Θ by putting additional bounds on $\|Dr\|_0$, $\|Dk_u\|_0$ and $\|Dk_s\|_0$. To this end, we define

$$\Gamma_0 := \left\{ \Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in C_b^1(X_c, X) \;\middle|\; \begin{array}{c} \Lambda(0) = 0,\; D\Lambda(0) = 0 \\ \text{and} \left\{ \begin{array}{l} \|Dr\|_0 \leq L_r \\ \|Dk_u\|_0 \leq L_u \\ \|Dk_s\|_0 \leq L_s \end{array} \right. \end{array} \right\}.$$

Theorem 3.3. *Assume that L_g and L_c are small in the sense of Remark 2.4 for $n = 2$. Then Θ is well-defined on Γ_0. Furthermore, Γ_0 is non-empty, invariant under Θ, and closed.*

Proof. We first note that by Remark 2.4 it holds that $0 \in \Gamma_0$ as L_r, L_u and L_s are all positive. Furthermore, from Remark 2.4 it follows that $L_r < \|A_c^{-1}\|_{\mathrm{op}}^{-1}$. By Section 3.2, we thus have that $A_c + r$ is a global diffeomorphism for $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$, hence Θ is well defined on Γ_0. Finally, it follows directly from the definition of Γ_0 that Γ_0 is closed. All that remains to prove is that Γ_0 is invariant under Θ.

Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$. Then we must prove that $\Theta(\Lambda) = \begin{pmatrix} \Theta_1(\Lambda) \\ \Theta_2(\Lambda) \\ \Theta_3(\Lambda) \end{pmatrix} \in \Gamma_0$. We will prove the resulting conditions on $\Theta(\Lambda)$ component-wise. We start by showing $\Theta(\Lambda)(0) = 0$. We have

$$\Theta_1(\Lambda)(0) = A_c k_c(0) + g_c(K(0)) - k_c((A_c + r)(0)).$$

With $\Lambda \in \Gamma_0$, we have $r(0) = 0$, thus $(A_c + r)(0) = 0$. So we get

$$\Theta_1(\Lambda)(0) = A_c k_c(0) + g_c(K(0)) - k_c(0).$$

By Assumption 4 of Theorem 2.1 we have $k_c(0) = 0$. Together with $k_u(0) = k_s(0) = 0$ we find $K(0) = \iota(0) + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}(0) = 0$. Therefore

$$\Theta_1(\Lambda)(0) = g_c(0).$$

Finally, again by Assumption 4 of Theorem 2.1, we have $g_c(0) = 0$ and we conclude that

$$\Theta_1(\Lambda)(0) = 0.$$

Similarly, $\Theta_2(\Lambda)(0) = 0$ and $\Theta_3(\Lambda)(0) = 0$. For the latter equality, we note that $A_c + r$ is a diffeomorphism and $(A_c + r)(0) = 0$, thus $(A_c + r)^{-1}(0) = 0$. So we find that $\Theta(\Lambda)(0) = 0$.

Likewise, we show that $D[\Theta(\Lambda)](0) = 0$ component-wise. For example, we have

$$\begin{aligned} D[\Theta_1(\Lambda)](0) &= A_c Dk_c(0) + Dg_c(K(0))DK(0) - Dk_c((A_c + r)(0))D(A_c + r)(0) \\ &= A_c Dk_c(0) + Dg_c(0)DK(0) - Dk_c(0)(A_c + Dr)(0). \end{aligned}$$

By Assumption 4 of Theorem 2.1 we have $Dg_c(0) = 0$ and $Dk_c(0) = 0$. So we find $D[\Theta_1(\Lambda)](0) = 0$. Similarly, we find $D[\Theta_2(\Lambda)](0) = 0$ and $D[\Theta_3(\Lambda)](0) = 0$. So we see that $D[\Theta(\Lambda)](0) = 0$.

For the bounds on the derivatives of the components of $\Theta(\Lambda)$, we use the shorthand notation $R = A_c + r$. For the first component we have

$$\|D[\Theta_1(\Lambda)]\|_0 \le \|D[A_c k_c]\|_0 + \|D[g_c \circ K]\|_0 + \|D[k_c \circ R]\|_0. \tag{3.3.1}$$

We have by Assumption 4 of Theorem 2.1

$$\|D[A_c k_c]\|_0 \le \|A_c\|_{\text{op}} \|Dk_c\|_0 \le \|A_c\|_{\text{op}} L_c. \tag{3.3.2}$$

Furthermore, we may estimate

$$\|DK\|_0 = \left\| \begin{pmatrix} D(\text{Id} + k_c) \\ Dk_u \\ Dk_s \end{pmatrix} \right\|_0 \le \max\{1 + \|Dk_c\|_0, \|Dk_u\|_0, \|Dk_s\|_0\}.$$

Now note that $1 + \|Dk_c\|_0 \le 1 + L_c$, $\|Dk_u\|_0 \le L_u$ and $\|Dk_s\|_0 \le L_s$. From the definition of L_u in (2.0.2c), we see that $L_u \le 1 + L_c$ if

$$\|A_u^{-1}\|_{\text{op}} L_g \le 1 - L_r \|A_u^{-1}\|_{\text{op}} - \|A_c\|_{\text{op}} \|A_u^{-1}\|_{\text{op}}.$$

The latter inequality holds since $L_u \ge 0$ and it is assumed that $\theta_{1,2} < 1$. This holds as $0 \le L_u$ and $\theta_{1,2} < 1$. Likewise, we can bound $L_s \le 1 + L_c$ as $\theta_{1,3} < 1$. Thus we find

$$\|DK\|_0 \le 1 + L_c. \tag{3.3.3}$$

With this estimate, we get

$$\|D[g_c \circ K]\|_0 \le \|Dg_c\|_0 \|DK\|_0 \le L_g(1+L_c). \tag{3.3.4}$$

Finally, we have

$$\|D[k_c \circ R]\|_0 \le \|Dk_c\|_0 \|DR\|_0 \le L_c \left(\|A_c\|_{\mathrm{op}} + L_r\right). \tag{3.3.5}$$

By combining (3.3.2), (3.3.4) and (3.3.5) we obtain from (3.3.1) that

$$\|D[\Theta_1(\Lambda)]\|_0 \le (1-L_c)\, L_r + L_c L_r = L_r.$$

The inequalities $\|D[\Theta_2(\Lambda)]\|_0 \le L_u$ and $\|D[\Theta_3(\Lambda)]\|_0 \le L_s$ follow from similar calculations.
Thus we see that Θ leaves Γ_0 invariant. □

3.4. A second invariant set for Θ

We would now like to prove that $\Theta : \Gamma_0 \to \Gamma_0$ is a contraction with respect to the C^1 norm, i.e. we must show that $\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_1 \le c\|\Lambda - \tilde{\Lambda}\|_1$ for all $\Lambda, \tilde{\Lambda} \in \Gamma_0$ and some $c < 1$. We will need to restrict to a subset of Γ_0 to obtain such a bound. To motivate our choice of the subset, let us look more carefully at the second component of Θ and focus on the bound on its derivative. For instance, one obtains an estimate of the form

$$\begin{aligned}\|D\Theta_2(\Lambda) - D\Theta_2(\tilde{\Lambda})\|_0 \le \|A_u^{-1}\|_{\mathrm{op}} \|D[k_u \circ (A_c + r)] - D[\tilde{k}_u \circ (A_c + \tilde{r})]\|_0 \\ + \|A_u^{-1}\|_{\mathrm{op}} \|D[g_u \circ K] - D[g_u \circ \tilde{K}]\|_0.\end{aligned}$$

If we now estimate the second factor of the first term, we obtain

$$\begin{aligned}\|D[k_u \circ (A_c + r)] - D[\tilde{k}_u \circ (A_c + \tilde{r})]\|_0 \le \|D[k_u \circ (A_c + r)] - D[k_u \circ (A_c + \tilde{r})]\|_0 \\ + \|D[k_u \circ (A_c + \tilde{r})] - D[\tilde{k}_u \circ (A_c + \tilde{r})]\|_0.\end{aligned}$$

We can bound the first term in the right hand side by

$$\begin{aligned}\|D[k_u \circ (A_c + r)] - D[k_u \circ (A_c + \tilde{r})]\|_0 \le \|Dk_u(A_c + r) - Dk_u(A_c + \tilde{r})\|_0 \|A_c\|_{\mathrm{op}} \\ + \|Dk_u(A_c + r)Dr - Dk_u(A_c + \tilde{r})D\tilde{r}\|_0.\end{aligned}$$

But, to estimate $\|Dk_u(A_c + r) - Dk_u(A_c + \tilde{r})\|_0$ in terms of $\|r - \tilde{r}\|_1$ we would need a uniform bound on the Lipschitz constant of Dk_u for $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$.

Therefore, we will restrict Θ to a subset of Γ_0 consisting of once differentiable functions with uniform bound on the Lipschitz constant of the derivative. In fact, we will restrict Θ to a subset of twice differentiable functions with uniform bound on the supremum norm of the second derivative. We make this choice rather than working with C^1 function with a Lipschitz bound on the derivative, because we later want to show that r, k_u and k_s are C^n. We thus define for $\delta > 0$ the space of C^2 functions

$$\Gamma_1(\delta) := \Gamma_0 \cap \left\{ \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in C_b^2(X_c, X) \,\middle|\, \begin{array}{l} \|D^2 r\|_0 \leq \delta \\ \|D^2 k_u\|_0 \leq \delta \\ \|D^2 k_s\|_0 \leq \delta \end{array} \right\}. \tag{3.4.1}$$

In Theorem 2.1, we assume that k_c and g have bounded second derivative. If we take $\|D^2 k_c\|_0 \leq \varepsilon$ and $\|D^2 g\|_0 \leq \varepsilon$ for some positive ε, then we will be able to construct an explicit $\delta(\varepsilon)$ such that $\Gamma_1(\delta(\varepsilon))$ is invariant under Θ.

Proposition 3.4. *Let $\varepsilon > 0$ and assume that $\|D^2 g\|_0, \|D^2 k_c\|_0 \leq \varepsilon$. Furthermore, assume that L_g and L_c are small in the sense of Remark 2.4 for $n = 2$. Then there exists a $\delta(\varepsilon) > 0$, which we explicitly define in (3.4.12), such that $\Gamma_1(\delta(\varepsilon))$ is invariant under Θ. Furthermore, $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$.*

Proof. Let $\varepsilon > 0$. We want to find $\delta(\varepsilon) > 0$ such that $\Gamma_1(\delta(\varepsilon))$ is invariant under Θ. Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_1(\delta)$, and introduce the shorthand notation $R = (A_c + r)$ and $T = (A_c + r)^{-1}$. We have to show that $\|D^2[\Theta_i(\Lambda)]\|_0 \leq \delta$ for $i = 1, 2, 3$.

We estimate the first component by

$$\|D^2 \Theta_1(\Lambda)\|_0 \leq \|D^2[A_c k_c]\|_0 + \|D^2[g_c \circ K]\|_0 + \|D^2[k_c \circ R]\|_0. \tag{3.4.2}$$

We estimate the terms separately. First,

$$\|D^2[A_c k_c]\|_0 = \|A_c D^2 k_c\|_0 \leq \|A_c\|_{\text{op}} \|D^2 k_c\|_0 \leq \|A_c\|_{\text{op}} \varepsilon. \tag{3.4.3}$$

Second, as $\|D^2 K\|_0 \leq \max\{\|D^2 k_c\|_0, \|D^2 k_u\|_0, \|D^2 k_s\|_0\} \leq \max\{\varepsilon, \delta\} \leq \varepsilon + \delta$, where we choose the rough estimate of $\varepsilon + \delta$ in order to write $\delta(\varepsilon)$ explicitly in (3.4.12), we find

$$\begin{aligned} \|D^2[g_c \circ K]\|_0 &\leq \|D^2 g_c(K)(DK, DK)\|_0 + \|Dg_c(K) D^2 K\|_0 \\ &\leq \|D^2 g_c\|_0 \|DK\|_0^2 + \|Dg_c\|_0 \|D^2 K\|_0 \\ &\leq (1 + L_c)^2 \varepsilon + L_g \varepsilon + L_g \delta. \end{aligned} \tag{3.4.4}$$

Finally, we have

$$\begin{aligned} \|D^2[k_c \circ R]\|_0 &\leq \|D^2 k_c(R)(DR, DR)\|_0 + \|Dk_c(R) D^2 r\|_0 \\ &\leq (\|A_c\|_{\text{op}} + L_r)^2 \varepsilon + L_c \delta. \end{aligned} \tag{3.4.5}$$

So inequality (3.4.2) together with estimates (3.4.3), (3.4.4) and (3.4.5) gives

$$\begin{aligned} \|D^2 \Theta_1(\Lambda)\|_0 &\leq \left(L_g + L_c\right) \delta + \left(\|A_c\|_{\text{op}} + (1 + L_c)^2 + L_g + (\|A_c\|_{\text{op}} + L_r)^2\right) \varepsilon \\ &= \theta_{2,1} \delta + C_1(\varepsilon), \end{aligned} \tag{3.4.6}$$

where $C_1(\varepsilon) := \left(\|A_c\|_{\text{op}} + (1 + L_c)^2 + L_g + (\|A_c\|_{\text{op}} + L_r)^2\right) \varepsilon$.

Likewise, we estimate the second component by

$$\|D^2[\Theta_2(\Lambda)]\|_0 \le \|A_u^{-1}\|_{\mathrm{op}}(\|D^2[g_u \circ K]\|_0 + \|D^2[k_u \circ R]\|_0). \tag{3.4.7}$$

As before, we estimate the terms separately to find

$$\begin{aligned}\|D^2[g_u \circ K]\|_0 &\le \|D^2 g_u\|_0\|DK\|_0^2 + \|Dg_u\|_0\|D^2K\|_0\\ &\le (1+L_c)^2\varepsilon + L_g\varepsilon + L_g\delta,\\ \|D^2[k_u \circ R]\|_0 &\le \|D^2k_u\|_0\|R\|_0^2 + \|Dk_u\|_0\|D^2r\|_0\\ &\le (\|A_c\|_{\mathrm{op}} + L_r)^2\delta + L_u\delta.\end{aligned}$$

Hence, inequality (3.4.7) becomes

$$\begin{aligned}\|D^2[\Theta_2(\Lambda)]\|_0 &\le \|A_u^{-1}\|_{\mathrm{op}}((\|A_c\|_{\mathrm{op}} + L_r)^2 + L_u + L_g)\delta + \|A_u^{-1}\|_{\mathrm{op}}((1+L_c)^2 + L_g)\varepsilon\\ &= \theta_{2,2}\delta + C_2(\varepsilon),\end{aligned} \tag{3.4.8}$$

where $C_2(\varepsilon) := \|A_u^{-1}\|_{\mathrm{op}}((1+L_c)^2 + L_g)\varepsilon$.

Finally, we estimate the third component by

$$\|D^2[\Theta_3(\Lambda)]\|_0 \le \|A_s\|_{\mathrm{op}}\|D^2[k_s \circ T]\|_0 + \|D^2[g_s \circ K \circ T]\|_0. \tag{3.4.9}$$

Before we estimate the different parts of the right hand side of (3.4.9), we estimate $\|D^2T\|_0$ by applying the chain rule twice to the right hand side of $0 = D^2\,\mathrm{Id} = D^2[R \circ T]$.

$$0 = D^2[R \circ T] = D^2R(T)(DT, DT) + DR(T)D^2T.$$

We let $DR(T)^{-1} = DT$ act on the left to obtain the upper bound

$$\|D^2T\|_0 = \|-DT\,D^2R(T)(DT, DT)\|_0 \le L_{-1}^3\delta. \tag{3.4.10}$$

We now find the following estimates for the terms in (3.4.9):

$$\begin{aligned}\|D^2[k_s \circ T]\|_0 &\le \|D^2k_s\|_0\|DT\|_0^2 + \|Dk_s\|_0\|D^2T\|_0\\ &\le L_{-1}^2\delta + L_sL_{-1}^3\delta,\\ \|D^2[g_s \circ K \circ T]\|_0 &\le \|D^2g_s\|_0(\|DK\|_0\|DT\|_0)^2 + \|Dg_s\|_0\|D^2K\|_0\|DT\|_0^2\\ &\quad + \|Dg_s\|_0\|DK\|_0\|D^2T\|_0\\ &\le L_{-1}^2(1+L_c)^2\varepsilon + L_{-1}^2L_g\delta + L_{-1}^2L_g\varepsilon + L_{-1}^3L_g(1+L_c)\delta.\end{aligned}$$

Inequality (3.4.9) thus becomes

$$\begin{aligned}\|D^3[\Theta_2(\Lambda)]\|_0 &\le L_{-1}^2(\|A_s\|_{\mathrm{op}}(1+L_{-1}L_s)+L_g(1+L_{-1}(1+L_c)))\delta\\ &\quad + L_{-1}^2((1+L_c)^2+L_g)\varepsilon\\ &= \theta_{2,3}\delta + C_3(\varepsilon),\end{aligned} \tag{3.4.11}$$

where $C_3(\varepsilon) := L_{-1}^2((1+L_c)^2+L_g)\varepsilon$.

If we show that $\delta(\varepsilon) > 0$ can be chosen such that

$$\theta_{2,i}\delta(\varepsilon) + C_i(\varepsilon) \le \delta(\varepsilon)$$

then we can estimate inequalities (3.4.6), (3.4.8) and (3.4.11) by $\delta(\varepsilon)$ which shows that $\Gamma_1(\delta(\varepsilon))$ is invariant under Θ. By assumption, we have $\theta_{2,i} < 1$ for $i = 1, 2, 3$. Therefore we can define

$$\delta(\varepsilon) := \max_{i=1,2,3}\left\{\frac{C_i(\varepsilon)}{1-\theta_{2,i}}\right\} > 0. \tag{3.4.12}$$

This gives for $i = 1, 2, 3$

$$\theta_{2,i}\delta(\varepsilon) + C_i(\varepsilon) = \theta_{2,i}\delta(\varepsilon) + (1-\theta_{2,i})\frac{C_i(\varepsilon)}{1-\theta_{2,1}} \le \theta_{2,i}\delta(\varepsilon) + (1-\theta_{2,i})\delta(\varepsilon) = \delta(\varepsilon).$$

Furthermore, we have by construction that $\delta(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$, since $C_i(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$ for $i = 1, 2, 3$. $\square$

3.5. Estimates for compositions

We already mentioned before Proposition 3.4 that we have to estimate expressions such as $\|D[k_u \circ (A_c + r)] - D[\tilde{k}_u \circ (A_c + \tilde{r})]\|_0$ in terms of $\|k_u - \tilde{k}_u\|_1$ and $\|r - \tilde{r}\|_1$ to show that Θ is a contraction with respect to the C^1 norm. Furthermore, we briefly showed part of the steps we would take to achieve the desired estimate. However, to show that Θ is a contraction with respect to the C^1 norm, we must also estimate expressions such as $k_u \circ (A_c + r) - \tilde{k}_u \circ (A_c + \tilde{r})$ in terms of $\|k_u - \tilde{k}_u\|_1$ and $\|r - \tilde{r}\|_1$. We will therefore provide a general result which allows use to bound both $\|k_u \circ (A_c + r) - \tilde{k}_u \circ (A_c + \tilde{r})\|_0$ and $\|D[k_u \circ (A_c + r)] - D[\tilde{k}_u \circ (A_c + \tilde{r})]\|_0$, and the corresponding expressions in the first and third component of $\Theta(\Lambda) - \Theta(\tilde{\Lambda})$. As we did in the definition of Γ_1, we prefer to work with twice differentiable functions instead of differentiable functions with Lipschitz first derivative.

Lemma 3.5. *Let X, Y and Z be Banach spaces.*

i) For $f_1 \in C_b^0(Y, Z)$, $g_1 \in C_b^1(Y, Z)$, $f_2, g_2 \in C_b^0(X, Y)$ we have the C^0-estimate

$$\|f_1 \circ f_2 - g_1 \circ g_2\|_0 \le \|f_1 - g_1\|_0 + \|Dg_1\|_0\|f_2 - g_2\|_0.$$

ii) For $f_1, g_1 \in C_b^2(Y, Z)$, $f_2, g_2 \in C_b^1(X, Y)$ we have the C^1-estimate

$$\begin{aligned}\|D[f_1 \circ f_2] - D[g_1 \circ g_2]\|_0 &\le \|Dg_2\|_0\left(\|D^2 g_1\|_0\|f_2 - g_2\|_0 + \|Df_1 - Dg_1\|_0\right)\\ &\quad + \|Df_1\|_0\|Df_2 - Dg_2\|_0.\end{aligned}$$

Proof. *i)* We find by the Mean Value Theorem

$$\begin{aligned}\|f_1\circ f_2-g_1\circ g_2\|_0 &\leq \|f_1\circ f_2-g_1\circ f_2\|_0+\|g_1\circ f_2-g_1\circ g_2\|_0\\ &\leq \|f_1\circ f_2-g_1\circ f_2\|_0+\left\|\int_0^1 Dg_1(tf_2+(1-t)g_2)\mathrm{d}t\,(f_2-g_2)\right\|_0\\ &\leq \|f_1-g_1\|_0+\|Dg_1\|_0\|f_2-g_2\|_0.\end{aligned}$$

ii) We use the chain rule and triangle inequality to find

$$\begin{aligned}\|D[f_1\circ f_2]-D[g_1\circ g_2]\|_0 &= \|Df_1(f_2)Df_2-Dg_1(g_2)Dg_2\|_0\\ &\leq \|Df_1(f_2)\,(Df_2-Dg_2)\,\|_0+\|\,(Df_1(f_2)-Dg_1(g_2))\,Dg_2\|_0.\end{aligned}$$

We estimate the first term using the submultiplicativity of the supremum norm:

$$\|Df_1(f_2)\,(Df_2-Dg_2)\,\|_0\leq \|Df_1\|_0\|Df_2-Dg_2\|_0.$$

For the second term, we again use submultiplicativity of the norm and estimate the factor $Df_1(f_2)-Dg_1(g_2)$ with part i) of this lemma:

$$\|(Df_1(f_2)-Dg_1(g_2))Dg_2\|_0\leq \Big(\|Df_1-Dg_1\|_0+\|D^2g_1\|_0\|f_2-g_2\|_0\Big)\,\|Dg_2\|_0. \quad \square$$

When we estimate the third component of $\Theta(\Lambda)-\Theta(\tilde{\Lambda})$ with the previous lemma, we get an estimate involving $\|(A_c+r)^{-1}-(A_c-\tilde{r})^{-1}\|_i$ instead of $\|r-\tilde{r}\|_i$ for $i=0,1$. So if we want to use the previous lemma to estimate $\Theta(\Lambda)-\Theta(\tilde{\Lambda})$ by $\|\Lambda-\tilde{\Lambda}\|_1$, then we must find an estimate for $\|(A_c+r)^{-1}-(A_c+\tilde{r})^{-1}\|_0$ and $\|D(A_c+r)^{-1}-D(A_c+\tilde{r})^{-1}\|_0$ in terms of $\|r-\tilde{r}\|_0$ and $\|Dr-D\tilde{r}\|_0$.

Lemma 3.6. *Let $r_1,r_2\in C_b^2(X_c,X_c)$ be such that A_c+r_i is a diffeomorphism with $D(A_c+r_i)^{-1}\in C_b^1(X_c,\mathcal{L}(X_c,X_c))$ for $i=1,2$.*

i) We have the C^0-estimate

$$\|(A_c+r_1)^{-1}-(A_c+r_2)^{-1}\|_0\leq \|D(A_c+r_1)^{-1}\|_0\|r_1-r_2\|_0.$$

ii) We have the C^1-estimate

$$\begin{aligned}&\|D(A_c+r_1)^{-1}-D(A_c+r_2)^{-1}\|_0\\ &\qquad\leq \|D(A_c+r_1)^{-1}\|_0\|D(A_c+r_2)^{-1}\|_0\Big(\|Dr_1-Dr_2\|_0\\ &\qquad\quad+\|D^2r_2\|_0\|D(A_c+r_1)^{-1}\|_0\|r_1-r_2\|_0\Big).\end{aligned}$$

Proof. *i)* We denote $R_i = A_c + r_i$ for $i = 1, 2$. Lemma 3.5i) implies that

$$\begin{aligned}\|R_1^{-1} - R_2^{-1}\|_0 &= \|R_1^{-1} \circ R_2 \circ R_2^{-1} - R_1^{-1} \circ R_1 \circ R_2^{-1}\|_0 \\ &\leq \|DR_1^{-1}\|_0 \|R_2 \circ R_2^{-1} - R_1 \circ R_2^{-1}\|_0 \\ &= \|DR_1^{-1}\|_0 \|r_1 - r_2\|_0.\end{aligned}$$

ii) For the C^1-estimate we denote $T_i = DR_i^{-1} \in C_b^1(X_c, \mathcal{L}(X_c, X_c))$ for $i = 1, 2$. Let $x \in X_c$, then we know that $T_i(x) \in \mathcal{L}(X_c, X_c)$ is invertible for $i = 1, 2$ by the Inverse Function Theorem. We find

$$\begin{aligned}\|T_1(x) - T_2(x)\|_{\mathrm{op}} &= \|T_2(x)\left(T_2(x)^{-1} - T_1(x)^{-1}\right)T_1(x)\|_{\mathrm{op}} \\ &\leq \|T_1(x)\|_{\mathrm{op}} \|T_2(x)\|_{\mathrm{op}} \|T_2(x)^{-1} - T_1(x)^{-1}\|_{\mathrm{op}}. \end{aligned} \tag{3.5.1}$$

Furthermore, the Inverse Function Theorem allows us to rewrite

$$T_i(x)^{-1} = \left(DR_i^{-1}(x)\right)^{-1} = DR_i(R_i^{-1}(x)) = A_c + Dr_i(R_i^{-1}(x)).$$

Taking the supremum over $x \in X_c$ gives

$$\|T_1^{-1} - T_2^{-1}\|_0 \leq \|Dr_1\left(R_1^{-1}\right) - Dr_2\left(R_2^{-1}\right)\|_0.$$

By using Lemma 3.5i) we find

$$\|T_1^{-1} - T_2^{-1}\|_0 \leq \|Dr_1 - Dr_2\|_0 + \|D^2 r_2\|_0 \|R_1^{-1} - R_2^{-1}\|_0.$$

Then we use part i) of this lemma applied to $\|R_1^{-1} - R_2^{-1}\|_0$ to obtain

$$\|T_1^{-1} - T_2^{-1}\|_0 \leq \|Dr_1 - Dr_2\|_0 + \|D^2 r_2\|_0 \|D(A_c + r_1)^{-1}\|_0 \|r_1 - r_2\|_0.$$

The result now follows from taking the supremum of $x \in X_c$ in inequality (3.5.1) and using the above estimate to bound $\sup_{x \in X_c} \|T_2(x)^{-1} - T_1(x)^{-1}\|_{\mathrm{op}}$. $\square$

To bound $\Theta(\Lambda) - \Theta(\tilde{\Lambda})$ by $\|\Lambda - \tilde{\Lambda}\|_1$, we can use Lemma 3.5 for the first two components. For the third component, we can use the same lemma together with Lemma 3.6 for the desired bound. However, while the first and second component consists of a single composition, the third component contains a double composition.

Lemma 3.7. *Let $r, \tilde{r} \in C_b^2(X_c, X_c)$ such that $\|Dr\|_0, \|D\tilde{r}\|_0 \leq L_r$ and assume that $L_r \leq \|A_c^{-1}\|_{\mathrm{op}}^{-1}$. Furthermore, let X and Y be Banach spaces with functions $h \in C_b^2(Y, X)$ and $f_1, f_2 \in C_b^2(X_c, Y)$.*

i) We have the C^0-estimate

$$\begin{aligned}&\|h \circ f_1 \circ (A_c + r)^{-1} - h \circ f_2 \circ (A_c + \tilde{r})^{-1}\|_0 \\ &\qquad \le \|Dh\|_0 \left(\|f_1 - f_2\|_0 + L_{-1}\|Df_2\|_0\|r - \tilde{r}\|_0\right).\end{aligned}$$

ii) If $\|D^2\tilde{r}\|_0 \le \delta(\varepsilon)$ for some $\varepsilon > 0$, then we have the C^1-estimate

$$\begin{aligned}&\|D[h \circ f_1 \circ (A_c + r)^{-1}] - D[h \circ f_2 \circ (A_c + \tilde{r})^{-1}]\|_0 \\ &\quad \le L_{-1}\left(\|Dh\|_0 + \|D^2h\|_0\|Df_2\|_0\right)\|f_1 - f_2\|_1 \\ &\qquad + L_{-1}^2\left(\|D^2h\|_0\|Df_2\|_0^2 + \|Dh\|_0\|D^2f_2\|_0\right)\|r - \tilde{r}\|_1 \\ &\qquad + L_{-1}^2\|Dh\|_0\|Df_1\|_0\left(1 + L_{-1}\delta(\varepsilon)\right)\|r - \tilde{r}\|_1.\end{aligned}$$

Proof. *i)* For the C^0-estimate, we first use Lemma 3.5i) twice:

$$\begin{aligned}&\|h \circ \left(f_1 \circ (A_c + r)^{-1}\right) - h \circ \left(f_2 \circ (A_c + \tilde{r})^{-1}\right)\|_0 \\ &\quad \le \|h - h\|_0 + \|Dh\|_0\|f_1 \circ (A_c + r)^{-1} - f_2 \circ (A_c + \tilde{r})^{-1}\|_0 \\ &\quad \le \|Dh\|_0\left(\|f_1 - f_2\|_0 + \|Df_2\|_0\|(A_c + r)^{-1} - (A_c + \tilde{r})^{-1}\|_0\right).\end{aligned}$$

Then we use Lemma 3.6i) to estimate $(A_c + r)^{-1} - (A_c + \tilde{r})^{-1}$:

$$\begin{aligned}&\|h \circ \left(f_1 \circ (A_c + r)^{-1}\right) - h \circ \left(f_2 \circ (A_c + \tilde{r})^{-1}\right)\|_0 \\ &\quad \le \|Dh\|_0\left(\|f_1 - f_2\|_0 + \|Df_2\|_0\|D(A_c + r)^{-1}\|_0\|r - \tilde{r}\|_0\right).\end{aligned}$$

Recall from Remark 2.4 that $L_r < \|A_c^{-1}\|_{\text{op}}^{-1}$ implies that $\|D(A_c + r)^{-1}\|_0 \le L_{-1}$, which proves the first estimate.

ii) To prove the C^1-estimate, we follow the same steps using Lemma 3.5ii) and Lemma 3.6ii) instead of Lemma 3.5i) and Lemma 3.6i) respectively. We then use the estimates $\|D^2\tilde{r}\|_0 \le \delta(\varepsilon)$, $\|D^i f_1 - D^i f_2\|_0 \le \|f_1 - f_2\|_1$ and $\|D^i r - D^i\tilde{r}\|_0 \le \|r - \tilde{r}\|_1$ for $i = 1, 2$. From this, the desired estimate follows. □

Remark 3.8. The assumptions on r and f_1 can be weakened in part i). We only need the assumptions that $A_c + r$ is a homeomorphism and f_1 is continuous and bounded.

3.6. A contraction

Following our proof scheme for Theorem 2.1, which we described in Section 2.1, we want to show that our fixed point operator $\Theta : \Gamma_1(\delta(\varepsilon)) \to \Gamma_1(\delta(\varepsilon))$, which is defined in (3.1.3), has a C^1 fixed point. We note that in Theorem 3.9 we will impose an upper bound on the second derivatives of the nonlinearities $k_c : X_c \to X_c$ and $g : X \to X$, whereas we only assume boundedness of the second derivatives in Theorem 2.1. However, we will see in Lemma 6.1 that we can always find a scaling such that the second derivative is sufficiently small.

Theorem 3.9. *Assume that L_g and L_c are small in the sense of Remark 2.4 for $n = 2$. There exists an $\varepsilon_0 > 0$ such for all $\varepsilon < \varepsilon_0$ it holds that if $\|D^2 g\|_0, \|D^2 k_c\|_0 \le \varepsilon$, then $\Theta : \Gamma_1(\delta(\varepsilon)) \to \Gamma_1(\delta(\varepsilon))$ is a contraction with respect to the C^1 norm.*

Proof. Let $\varepsilon > 0$ and $\|D^2 g\|_0, \|D^2 k_c\|_0 \le \varepsilon$. Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix}$, $\tilde{\Lambda} = \begin{pmatrix} \tilde{r} \\ \tilde{k}_u \\ \tilde{k}_s \end{pmatrix} \in \Gamma_1(\delta(\varepsilon))$. We denote $R = A_c + r$ and $\tilde{R} = A_c + \tilde{r}$.

Our proof that Θ is a C^1-contraction is divided in three steps.

A) We prove that Θ is a contraction with respect to the C^0 norm, independent of ε.
B) We show the existence of a constant $\theta_1(\varepsilon)$ such that

$$\|D[\Theta(\Lambda)] - D[\Theta(\tilde{\Lambda})]\|_0 \le \theta_1(\varepsilon)\|\Lambda - \tilde{\Lambda}\|_1.$$

C) We show that $\varepsilon > 0$ can be chosen so that $\theta_1(\varepsilon) < 1$, thus proving that Θ is a contraction with respect to the C^1 norm.

Step A) We want to find $\theta_0 < 1$ such that

$$\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_0 \le \theta_0\|\Lambda - \tilde{\Lambda}\|_0.$$

Recall from equation (3.1.3) that

$$\Theta(\Lambda) = \Theta\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} = \begin{pmatrix} A_c k_c + g_c \circ K - k_c \circ (A_c + r) \\ A_u^{-1} k_u \circ (A_c + r) - A_u^{-1} g_u \circ K \\ A_s k_s \circ (A_c + r)^{-1} + g_s \circ K \circ (A_c + r)^{-1} \end{pmatrix}.$$

We will find the contraction constant component-wise, i.e. we will show that

$$\|\Theta_i(\Lambda) - \Theta_i(\tilde{\Lambda})\|_0 \le \theta_{0,i}\|\Lambda - \tilde{\Lambda}\|_0$$

with $\theta_{0,i}$ given explicitly in equation (2.0.3a) to (2.0.3c) for $i = 1, 2, 3$.

r-component: We start with

$$\|\Theta_1(\Lambda) - \Theta_1(\tilde{\Lambda})\|_0 \le \|g_c \circ K - g_c \circ \tilde{K}\|_0 + \|k_c \circ R - k_c \circ \tilde{R}\|_0. \tag{3.6.1}$$

By using Lemma 3.5i) we find the estimate

$$\|g_c \circ K - g_c \circ \tilde{K}\|_0 \le \|Dg_c\|_0\|K - \tilde{K}\|_0 \le L_g\|K - \tilde{K}\|_0 \le L_g\|\Lambda - \tilde{\Lambda}\|_0. \tag{3.6.2}$$

Here we recall that $\|Dg_c\|_0 \le L_g$, which follows from Assumption 4 of Theorem 2.1. Likewise, we estimate

$$\|k_c \circ R - k_c \circ \tilde{R}\|_0 \le \|Dk_c\|_0\|R - \tilde{R}\|_0 = \|Dk_c\|_0\|r - \tilde{r}\|_0 \le L_c\|\Lambda - \tilde{\Lambda}\|_0. \tag{3.6.3}$$

Thus inequality (3.6.1) together with estimates (3.6.2) and (3.6.3) gives

$$\|\Theta_1(\Lambda) - \Theta_1(\tilde{\Lambda})\|_0 \leq \left(L_g + L_c\right) \|\Lambda - \tilde{\Lambda}\|_0 = \theta_{0,1}\|\Lambda - \tilde{\Lambda}\|_0. \tag{3.6.4}$$

<u>*k_u-component:*</u> Similarly, we have

$$\begin{aligned}\|\Theta_2(\Lambda) - \Theta_2(\tilde{\Lambda})\|_0 &\leq \|A_u^{-1}\left(k_u \circ R - \tilde{k}_u \circ \tilde{R}\right)\|_0 + \|A_u^{-1}\left(g_u \circ K - g_u \circ \tilde{K}\right)\|_0 \\ &\leq \|A_u^{-1}\|_{\mathrm{op}}\left(\|k_u \circ R - \tilde{k}_u \circ \tilde{R}\|_0 + \|g_u \circ K - g_u \circ \tilde{K}\|_0\right). \end{aligned}\tag{3.6.5}$$

We again use Lemma 3.5i), which gives

$$\begin{aligned}&\|g_u \circ K - g_u \circ \tilde{K}\|_0 \leq \|Dg_u\|_0\|K - \tilde{K}\|_0 \leq L_g\|\Lambda - \tilde{\Lambda}\|_0, \\ &\|k_u \circ R - \tilde{k}_u \circ \tilde{R}\|_0 \leq \|k_u - \tilde{k}_u\|_0 + \|D\tilde{k}_u\|_0\|R - \tilde{R}\|_0 \leq (1 + L_u)\|\Lambda - \tilde{\Lambda}\|_0.\end{aligned}$$

Here we used that $\|D\tilde{k}_u\|_0 \leq L_u$, which follows from the fact that $\Gamma_1(\delta(\varepsilon)) \subset \Gamma_0$, with the latter space defined before Theorem 3.3. Thus inequality (3.6.5) becomes

$$\|\Theta_2(\Lambda) - \Theta_2(\tilde{\Lambda})\|_0 \leq \|A_u^{-1}\|_{\mathrm{op}}\left(1 + L_u + L_g\right)\|\Lambda - \tilde{\Lambda}\|_0 = \theta_{0,2}\|\Lambda - \tilde{\Lambda}\|_0. \tag{3.6.6}$$

<u>*k_s-component:*</u> Let $T = (A_c + r)^{-1}$ and $\tilde{T} = (A_c + \tilde{r})^{-1}$, then we have

$$\|\Theta_3(\Lambda) - \Theta_3(\tilde{\Lambda})\|_0 \leq \|A_s \circ k_s \circ T - A_c \circ \tilde{k}_s \circ \tilde{T}\|_0 + \|g_s \circ K \circ T - g_s \circ \tilde{K} \circ \tilde{T}\|_0. \tag{3.6.7}$$

We use Lemma 3.7i), where the condition $L_r < \|A_c^{-1}\|_{\mathrm{op}}^{-1}$ is satisfied by Remark 2.4, to obtain

$$\begin{aligned}\|A_s \circ k_s \circ T - A_c \circ \tilde{k}_s \circ \tilde{T}\|_0 &\leq \|A_c\|_{\mathrm{op}}\left(\|k_s - \tilde{k}_s\|_0 + L_{-1}\|D\tilde{k}_s\|_0\|r - \tilde{r}\|_0\right) \\ &\leq \|A_c\|_{\mathrm{op}}\left(1 + L_{-1}L_s\right)\|\Lambda - \tilde{\Lambda}\|_0, \\ \|g_s \circ K \circ T - g_s \circ \tilde{K} \circ \tilde{T}\|_0 &\leq \|Dg_s\|_0\left(\|K - \tilde{K}\|_0 + L_{-1}\|D\tilde{K}\|_0\|r - \tilde{r}\|_0\right) \\ &\leq L_g\left(1 + L_{-1}(1 + L_c)\right)\|\Lambda - \tilde{\Lambda}\|_0.\end{aligned}$$

We used $\|D\tilde{K}\|_0 \leq 1 + L_c$, see (3.3.3), in the last estimate. Thus inequality (3.6.7) becomes

$$\begin{aligned}\|\Theta_3(\Lambda) - \Theta_3(\tilde{\Lambda})\|_0 &\leq \left(\|A_s\|_{\mathrm{op}}\left(1 + L_s L_{-1}\right) + L_g\left(1 + L_{-1}\left(1 + L_c\right)\right)\right)\|\Lambda - \tilde{\Lambda}\|_0 \\ &= \theta_{0,3}\|\Lambda - \tilde{\Lambda}\|_0.\end{aligned}\tag{3.6.8}$$

<u>*Contraction constant:*</u> We can now estimate $\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_0$ with inequalities (3.6.4), (3.6.6) and (3.6.8). We obtain

$$\begin{aligned}\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_0 &= \max_{i=1,2,3}\left\{\|\Theta_i(\Lambda) - \Theta_i(\tilde{\Lambda})\|_0\right\} \\ &\leq \max_{i=1,2,3}\left\{\theta_{0,i}\|\Lambda - \tilde{\Lambda}\|_0\right\} \\ &= \theta_0\|\Lambda - \tilde{\Lambda}\|_0.\end{aligned}\tag{3.6.9}$$

Here we define $\theta_0 := \max\{\theta_{0,1}, \theta_{0,2}, \theta_{0,3}\}$. Since it is assumed that Remark 2.4 holds for $n = 2$, we have $\theta_{0,i} < 1$ and thus $\theta_0 < 1$. This implies that Θ is a contraction with respect to the C^0 norm.

Step B) Analogous to step A), we want to prove the component-wise inequality

$$\|D[\Theta_i(\Lambda)] - D[\Theta_i(\tilde{\Lambda})]\|_0 \leq \left(\theta_{1,i} + C_{1,i}(\varepsilon)\right) \|\Lambda - \tilde{\Lambda}\|_1,$$

with $\theta_{1,i}$ defined in (2.0.3a) to (2.0.3c) and $C_{1,i}$ defined below in the proof. We note that $\Lambda, \tilde{\Lambda} \in \Gamma_1(\delta(\varepsilon))$, so we have $\|D^2 r\|_0, \|D^2 k_u\|_0, \|D^2 k_s\|_0 \leq \delta(\varepsilon)$.

r-component: We start with

$$\begin{aligned} &\|D[\Theta_1(\Lambda)] - D[\Theta_1(\tilde{\Lambda})]\|_0 \\ &\quad \leq \|D[g_c \circ K] - D[g_c \circ \tilde{K}]\|_0 + \|D[k_c \circ R] - D[k_c \circ \tilde{R}]\|_0. \end{aligned} \tag{3.6.10}$$

We infer from Lemma 3.5ii) that

$$\begin{aligned} \|D[g_c \circ K] - D[g_c \circ \tilde{K}]\|_0 &\leq \|D\tilde{K}\|_0 \|D^2 g_c\|_0 \|K - \tilde{K}\|_0 + \|Dg_c\|_0 \|DK - D\tilde{K}\|_0 \\ &\leq \left(\|D\tilde{K}\|_0 \|D^2 g_c\|_0 + \|Dg_c\|_0\right) \|\Lambda - \tilde{\Lambda}\|_1 \\ &\leq \left((1 + L_c)\,\varepsilon + L_g\right) \|\Lambda - \tilde{\Lambda}\|_1, \end{aligned} \tag{3.6.11}$$

where we have used (3.3.3). Likewise, we find the estimate

$$\begin{aligned} \|D[k_c \circ R] - D[k_c \circ \tilde{R}]\|_0 &\leq \|D\tilde{R}\|_0 \|D^2 k_c\|_0 \|r - \tilde{r}\|_0 + \|Dk_c\|_0 \|Dr - D\tilde{r}\|_0 \\ &\leq \left(\left(\|A_c\|_{\mathrm{op}} + L_r\right)\varepsilon + L_c\right) \|\Lambda - \tilde{\Lambda}\|_1. \end{aligned} \tag{3.6.12}$$

Thus inequality (3.6.10) together with estimates (3.6.11) and (3.6.12) gives

$$\begin{aligned} \|D[\Theta_1(\Lambda)] - D[\Theta_1(\tilde{\Lambda})]\|_0 &\leq \left(L_g + L_c + \left(1 + L_c + \|A_c\|_{\mathrm{op}} + L_r\right)\varepsilon\right) \|\Lambda - \tilde{\Lambda}\|_1 \\ &= \left(\theta_{1,1} + C_{1,1}(\varepsilon)\right) \|\Lambda - \tilde{\Lambda}\|_1, \end{aligned} \tag{3.6.13}$$

where we define $C_{1,1}(\varepsilon) := \left(1 + L_c + \|A_c\|_{\mathrm{op}} + L_r\right)\varepsilon$.

k_u-component: Similarly, we have

$$\begin{aligned} \|D[\Theta_2(\Lambda)] - D[\Theta_2(\tilde{\Lambda})]\|_0 &\leq \|A_u^{-1}\|_{\mathrm{op}} \|D[g_u \circ K] - D[g_u \circ \tilde{K}]\|_0 \\ &\quad + \|A_u^{-1}\|_{\mathrm{op}} \|D[k_u \circ R] - D[\tilde{k}_u \circ \tilde{R}]\|_0. \end{aligned} \tag{3.6.14}$$

Using Lemma 3.5ii) we get

$$\begin{aligned} \|D[g_u \circ K] - D[g_u \circ \tilde{K}]\|_0 &\leq \|D\tilde{K}\|_0 \|D^2 g_u\|_0 \|K - \tilde{K}\|_0 + \|Dg_u\|_0 \|DK - D\tilde{K}\|_0 \\ &\leq \left((1 + L_c)\,\varepsilon + L_g\right) \|\Lambda - \tilde{\Lambda}\|_1, \\ \|D[k_u \circ R] - D[\tilde{k}_u \circ \tilde{R}]\|_0 &\leq \|D\tilde{R}\|_0 \left(\|D^2 \tilde{k}_u\|_0 \|r - \tilde{r}\|_0 + \|Dk_u - D\tilde{k}_u\|_0\right) \end{aligned}$$

$$
\begin{aligned}
&+ \|Dk_u\|_0 \|Dr - D\tilde{r}\|_0 \\
&\leq \left(\left(\|A_c\|_{\mathrm{op}} + L_r\right)(1+\delta(\varepsilon)) + L_u\right) \|\Lambda - \tilde{\Lambda}\|_1.
\end{aligned}
$$

Thus inequality (3.6.14) becomes

$$
\|D\Theta_2(\Lambda) - D\Theta_2(\tilde{\Lambda})\|_0 \leq \left(\theta_{1,2} + C_{1,2}(\varepsilon)\right) \|\Lambda - \tilde{\Lambda}\|_1, \tag{3.6.15}
$$

where we define $C_{1,2}(\varepsilon) := \|A_u^{-1}\|_{\mathrm{op}} \left(L_g(1+L_c)\varepsilon + (\|A_c\|_{\mathrm{op}} + L_r)\delta(\varepsilon)\right)$.

k_s*-component:* Recall that $T = (A_c + r)^{-1}$ and $\tilde{T} = (A_c + \tilde{r})^{-1}$, then we have

$$
\begin{aligned}
\|D[\Theta_3(\Lambda)] - D[\Theta_3(\tilde{\Lambda})]\|_0 \leq{}& \|D[A_s \circ k_s \circ T] - D[A_s \circ \tilde{k}_s \circ \tilde{T}]\|_0 \\
&+ \|D[g_s \circ K \circ T] - D[g_s \circ \tilde{K} \circ \tilde{T}]\|_0.
\end{aligned} \tag{3.6.16}
$$

We will estimate both terms with Lemma 3.7ii). For the first term, we note that $\|DA_s\|_0 = \|A_s\|_{\mathrm{op}}$ and $\|D^2 A_s\|_0 = 0$, which gives us

$$
\begin{aligned}
&\|D[A_s \circ k_s \circ T] - D[A_s \circ \tilde{k}_s \circ \tilde{T}]\|_0 \\
&\quad \leq L_{-1}\|A_s\|_{\mathrm{op}}\|k_s - \tilde{k}_s\|_1 + L_{-1}^2 \|A_s\|_{\mathrm{op}} \|D^2\tilde{k}_s\|_0 \|r - \tilde{r}\|_1 \\
&\qquad + L_{-1}^2 \|A_s\|_{\mathrm{op}} \|Dk_s\|_0 (1 + L_{-1}\delta(\varepsilon)) \|r - \tilde{r}\|_1 \\
&\quad \leq \|A_s\|_{\mathrm{op}} L_{-1}\left(1 + L_{-1}L_s\right) \|\Lambda - \tilde{\Lambda}\|_1 \qquad (3.6.17) \\
&\qquad + \|A_s\|_{\mathrm{op}} L_{-1}^2 \delta(\varepsilon)\left(1 + L_{-1}L_s\right) \|\Lambda - \tilde{\Lambda}\|_1, \qquad (3.6.18)
\end{aligned}
$$

where we grouped the terms with and without a factor $\delta(\varepsilon)$. The second term in (3.6.16) involves the first and second derivative of $\tilde{K}$. We estimate the first derivate again with $1 + L_c$ and we estimate the second derivative with $\|D^2\tilde{K}\|_0 = \max\{\|D^2 k_c\|_0, \|D^2\tilde{k}_u\|_0, \|D^2\tilde{k}_s\|_0\} \leq \max\{\varepsilon, \delta(\varepsilon)\} =: \gamma(\varepsilon)$. Hence we obtain

$$
\begin{aligned}
&\|D[g_s \circ K \circ T] - D[g_s \circ \tilde{K} \circ \tilde{T}]\|_0 \\
&\quad \leq L_{-1}\left(\|Dg_s\|_0 + \|D^2 g_s\|_0 \|D\tilde{K}\|_0\right) \|K - \tilde{K}\|_1 \\
&\qquad + L_{-1}^2 \left(\|D^2 g_s\|_0 \|D\tilde{K}\|_0^2 + \|Dg_s\|_0 \|D^2\tilde{K}\|_0\right) \|r - \tilde{r}\|_1 \\
&\qquad + L_{-1}^2 \|Dg_s\|_0 \|DK\|_0 (1 + L_{-1}\delta(\varepsilon)) \|r - \tilde{r}\|_1 \\
&\quad \leq L_{-1}\left(L_g + L_{-1}L_g(1+L_c)\right) \|\Lambda - \tilde{\Lambda}\|_1 \qquad (3.6.19) \\
&\qquad + L_{-1}\left((1+L_c)\varepsilon + L_{-1}(1+L_c)^2\varepsilon\right) \|\Lambda - \tilde{\Lambda}\|_1 \\
&\qquad + L_{-1}^2\left(L_g\gamma(\varepsilon) + L_{-1}L_g(1+L_c)\delta(\varepsilon)\right) \|\Lambda - \tilde{\Lambda}\|_1. \qquad (3.6.20)
\end{aligned}
$$

Here we again grouped the terms with and without ε. We see that (3.6.17) and (3.6.19) together are $\theta_{1,3}\|\Lambda - \tilde{\Lambda}\|_1$. Likewise, we can estimate (3.6.18) and (3.6.20) together by $\theta_{2,3}\gamma(\varepsilon)\|\Lambda - \tilde{\Lambda}\|_1$ as $\delta(\varepsilon) \leq \gamma(\varepsilon)$. Then inequality (3.6.16) reduces to

$$\|D\Theta_3(\Lambda) - D\Theta_3(\tilde{\Lambda})\|_0 \leq \left(\theta_{1,3} + C_{1,3}(\varepsilon)\right) \|\Lambda - \tilde{\Lambda}\|_1, \tag{3.6.21}$$

where we define $C_{1,3}(\varepsilon) := \theta_{2,3}\gamma(\varepsilon) + L_{-1}(1+L_c)\varepsilon + L_{-1}^2(1+L_c)^2\varepsilon$.

Lipschitz constant: Inequalities (3.6.13), (3.6.15) and (3.6.21) give

$$\begin{aligned}
\|D[\Theta(\Lambda)] - D[\Theta(\tilde{\Lambda})]\|_0 &= \max_{i=1,2,3}\left\{\|D[\Theta_i(\Lambda)] - D[\Theta_i(\tilde{\Lambda})]\|_0\right\} \\
&\leq \max_{i=1,2,3}\left\{\left(\theta_{1,i} + C_{1,i}(\varepsilon)\right)\|\Lambda - \tilde{\Lambda}\|_1\right\} \\
&\leq \theta_1(\varepsilon)\|\Lambda - \tilde{\Lambda}\|_1.
\end{aligned} \tag{3.6.22}$$

Here we define

$$\theta_1(\varepsilon) := \max\left\{\theta_{1,1}, \theta_{1,2}, \theta_{1,3}\right\} + \max\{C_{1,1}(\varepsilon), C_{1,2}(\varepsilon), C_{1,3}(\varepsilon)\}. \tag{3.6.23}$$

Step C) From Remark 2.4 it follows that

$$\max\left\{\theta_{1,1}, \theta_{1,2}, \theta_{1,3}\right\} < 1.$$

As $\delta(\varepsilon) \downarrow 0$ and thus also $\gamma(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$, see Proposition 3.4, we have

$$\lim_{\varepsilon \to 0} \max\{C_{1,1}(\varepsilon), C_{1,2}(\varepsilon), C_{1,3}(\varepsilon)\} = 0.$$

We infer that

$$\lim_{\varepsilon \to 0} \theta_1(\varepsilon) = \max\left\{\theta_{1,1}, \theta_{1,2}, \theta_{1,3}\right\} < 1.$$

Hence, we can find an $\varepsilon_0 > 0$ such that $\theta_1(\varepsilon) < 1$ for all $\varepsilon < \varepsilon_0$. Then estimates (3.6.9) and (3.6.22) imply that

$$\begin{aligned}
\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_1 &= \max\left\{\|\Theta(\Lambda) - \Theta(\tilde{\Lambda})\|_0, \|D[\Theta(\Lambda)] - D[\Theta(\tilde{\Lambda})]\|_0\right\} \\
&\leq \max\left\{\theta_0\|\Lambda - \tilde{\Lambda}\|_0, \theta_1(\varepsilon)\|\Lambda - \tilde{\Lambda}\|_1\right\} \\
&\leq \lambda_1\|\Lambda - \tilde{\Lambda}\|_1.
\end{aligned}$$

We define the contraction constant $\lambda_1 := \max\{\theta_0, \theta_1(\varepsilon)\}$, which is smaller than 1 for $\varepsilon < \varepsilon_0$ by our previous discussion. We conclude that $\Theta : \Gamma_1(\delta(\varepsilon)) \to \Gamma_1(\delta(\varepsilon))$ is a contraction with respect to the C^1 norm for $\varepsilon < \varepsilon_0$. □

We can now prove the existence of a C^1 center manifold under the assumption that the second derivative of k_c and g are small enough. As we will see in Lemma 6.1, we can always find a scaling such that these second derivatives will be sufficiently small.

Corollary 3.10. *Let $\varepsilon > 0$ be such that $F : X \to X$ satisfies the assumptions of Theorems 2.1 and 3.9. Then the conclusion of Theorem 2.1 holds for $K \in C^1(X_c, X)$ and $r \in C^1(X_c, X_c)$. In particular, the image of K is a C^1 center manifold for F.*

Proof. By assumption, $\varepsilon > 0$ is such that Θ is a contraction. In Theorem 3.9 and Proposition 3.1 we proved the existence of a conjugacy K and conjugate dynamics $A_c + r$. Furthermore, from the definition of Γ_0, it follows that K and r satisfy the Properties B) and A) respectively. In particular, it follows that image of K is invariant under F and tangent to X_c at 0, hence the image of K is a C^1 center manifold for F. $\square$

4. A C^2 center manifold

Now that we have a C^1 conjugacy, the third step in our proof scheme in Section 2.1 is showing that the conjugacy is C^2. We will prove the equivalent statement that the derivative of the C^1 conjugacy is also C^1. For this, we define another fixed point operator acting on C^1 functions, and show that its fixed point is the derivative of the conjugacy from Corollary 3.10.

4.1. A new fixed point operator

We first note that Θ is a contraction with respect to the C^1 norm on $\Gamma_1(\delta(\varepsilon))$, a set that is not closed with respect to the C^1 norm. That means that the fixed point of Θ lies in the C^1 closure of $\Gamma_1(\delta(\varepsilon))$, which is enclosed by Γ_0.

Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$ denote any fixed point of Θ, i.e. Λ consists of three C^1 functions and we have

$$\begin{pmatrix} r \\ k_s \\ k_s \end{pmatrix} = \Lambda = \Theta(\Lambda) = \begin{pmatrix} A_c k_c + g_c \circ K - k_c \circ (A_c + r) \\ A_u^{-1} k_u \circ (A_c + r) - A_u^{-1} g_u \circ K \\ A_s k_s \circ (A_c + r)^{-1} + g_s \circ K \circ (A_c + r)^{-1} \end{pmatrix}.$$

We can therefore take the derivative at both sides of the equation, which gives

$$\begin{pmatrix} Dr \\ Dk_u \\ Dk_s \end{pmatrix} = \begin{pmatrix} A_c Dk_c + Dg_c(K)DK - Dk_c(R)\,DR \\ -A_u^{-1} Dg_u(K)DK + A_u^{-1} Dk_u(R)DR \\ A_s Dk_s(T)DT + Dg_s(K \circ T)DK(T)DT \end{pmatrix}, \tag{4.1.1}$$

where we define $R := A_c + r$ and $T := (A_c + r)^{-1}$, notation that we will use throughout the rest of the paper. To express $DT = D(A_c + R)^{-1}$ in terms of r and Dr, we use the Inverse Function Theorem and write

$$DT(x) = D(A_c + r)^{-1}(x) = \left(DR(R^{-1}(x))\right)^{-1} = (DR(T(x)))^{-1}.$$

This motivates us to introduce for $\rho : X_c \to \mathcal{L}(X_c, X_c)$ the functions

$$\begin{aligned} P_\rho : X_c &\to \mathcal{L}(X_c, X_c) \\ x &\mapsto A_c + \rho(x) \end{aligned} \quad \text{and} \quad \begin{aligned} Q_\rho : X_c &\mapsto \mathcal{L}(X_c, X_c) \\ x &\mapsto (P_\rho(T(x)))^{-1} \end{aligned} \tag{4.1.2}$$

so that we can write $DT(x) = Q_{Dr}(x)$ and $DR(x) = P_{Dr}(x)$. In view of (4.1.1) we use these functions to introduce the fixed point operator

$$\Theta^{[2]} : \begin{pmatrix} \rho \\ \kappa_u \\ \kappa_s \end{pmatrix} \mapsto \begin{pmatrix} A_c Dk_c + Dg_c(K)\kappa - Dk_c\,(R)\,P_\rho \\ -A_u^{-1} Dg_u(K)\kappa + A_u^{-1}\kappa_u(R)P_\rho \\ A_s\kappa_s(T)Q_\rho + Dg_s(K\circ T)\kappa(T)Q_\rho \end{pmatrix} \tag{4.1.3}$$

where $\kappa = \begin{pmatrix} \mathrm{Id}+Dk_c \\ \kappa_u \\ \kappa_s \end{pmatrix}$ and $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$ is a fixed point of Θ. To summarize, we have the following proposition:

Proposition 4.1. *Let $\Lambda \in \Gamma_0$ be any fixed point of the operator Θ. Then $D\Lambda$ is a fixed point of the operator $\Theta^{[2]}$ defined in* (4.1.3).

Proof. This follows immediately from the above discussion. □

We want to use $\Theta^{[2]}$ to show that $D\Lambda$ is C^1 instead of only C^0. To this end, we want to show that $\Theta^{[2]}$ is a contraction in C^1 on a suitable set of C^1 functions, and show that its fixed point in this set is $D\Lambda$. We therefore want to restrict $\Theta^{[2]}$ to a space similar to $\Gamma_1(\delta(\varepsilon))$. In particular, we want to reflect that Θ is a fixed point operator acting on functions and $\Theta^{[2]}$ is a fixed point operator acting on derivatives. So where functions in $\Gamma_1(\delta(\varepsilon))$ have restrictions on the first and second derivative, we want the same restrictions on the function and its derivative in our new space respectively. Therefore, let $\delta > 0$, and define the set

$$\Gamma_2(\delta) := \left\{ \mathcal{M} = \begin{pmatrix} \rho \\ \kappa_u \\ \kappa_s \end{pmatrix} \in C_b^1\left(X_c, \mathcal{L}(X_c, X)\right) \;\middle|\; \begin{array}{l} \mathcal{M}(0) = 0, \\ \|\rho\|_0 \le L_r \\ \|\kappa_u\|_0 \le L_u \\ \|\kappa_s\|_0 \le L_s \\ \|D\mathcal{M}\|_0 \le \delta \end{array} \right\}. \tag{4.1.4}$$

Proposition 4.2. *Let $\varepsilon > 0$ and assume that $\|D^2 g\|_0, \|D^2 k_c\|_0 \le \varepsilon$. Furthermore, assume that L_g and L_c are small in the sense of Remark 2.4 for $n = 2$. Then, for $\delta(\varepsilon) > 0$ from Proposition 3.4, the set $\Gamma_2(\delta(\varepsilon))$ is invariant under $\Theta^{[2]}$.*

Proof. The proof follows from similar estimates as in Theorem 3.3 for the bounds on ρ, κ_u and κ_s as well as for $\mathcal{M}(0) = 0$. The bound on the second derivative follows from the same estimates as in Proposition 3.4. We will illustrate this for the derivative of the second component, i.e. we will show that $D\Theta_1^{[2]}(\mathcal{M})$ is bounded by $\delta(\varepsilon)$ for $\mathcal{M} \in \Gamma_2(\delta(\varepsilon))$.

We start as we did in Proposition 3.4 with

$$\|D\Theta_1^{[2]}(\mathcal{M})\|_0 \le \|D[A_c Dk_c]\|_0 + \|D[Dg_c(K)\kappa]\|_0 + \|D[Dk_c(R)P_\rho]\|_0.$$

We again estimate the terms separately:

$$\|A_c D^2 k_c\|_0 = \|A_c\|_{\mathrm{op}} \|Dk_c\|_0 \le \|A_c\|_{\mathrm{op}}\varepsilon,$$
$$\|D[Dg_c(K)\kappa]\|_0 \le (1 + L_c)^2\varepsilon + L_g(\varepsilon + \delta(\varepsilon)),$$

$$\|D[Dk_c(R)P]\|_0 \leq (\|A_c\|_{\mathrm{op}} + L_r)^2\varepsilon + L_c\delta(\varepsilon),$$

where we used $D\kappa \leq \varepsilon + \delta(\varepsilon)$ as we did in (3.4.4). All together, we find

$$\|D\Theta_1^{[2]}(\mathcal{M})\|_0 \leq \theta_{2,1}\delta(\varepsilon) + C_1(\varepsilon) \leq \delta(\varepsilon).$$

Here we used the definition of $C_1(\varepsilon)$ just below (3.4.8), and the last inequality follows from the definition of $\delta(\varepsilon)$ in (3.4.12). The other estimates are similar. □

4.2. *Estimates for products*

In Section 3.5 we gave some preliminary results for Theorem 3.9 in Lemmas 3.5 and 3.7. We want to derive similar results for derivatives instead of functions in Lemmas 4.3 and 4.4 respectively. The results below will be framed in a slightly more general setting, so that we can use them in the next section as well.

Lemma 4.3. *Let X, Y and Z be Banach spaces, $m \in \mathbb{N}$ and $h \in C_b^1(X, Y)$.*

i) For $f_1, g_1 \in C_b^0(Y, \mathcal{L}(Y, Z))$, $f_2, g_2 \in C_b^0(X, \mathcal{L}^m(X, Y))$ we have the C^0-estimate

$$\|(f_1 \circ h)f_2 - (g_1 \circ h)g_2\|_0 \leq \|f_1\|_0\|f_2 - g_2\|_0 + \|f_1 - g_1\|_0\|g_2\|_0.$$

ii) For $f_1, g_1 \in C_b^1(Y, \mathcal{L}(Y, Z))$, $f_2, g_2 \in C_b^1(X, \mathcal{L}^m(X, Y))$ we have the C^1-estimate

$$\begin{aligned}\|D[(f_1 \circ h)f_2] - D[(g_1 \circ h)g_2]\|_0 \\ &\leq \|Df_1\|_0\|Dh\|_0\|f_2 - g_2\|_0 + \|Df_1 - Dg_1\|_0\|Dh\|_0\|g_2\|_0 \\ &\quad + \|f_1\|_0\|Df_2 - Dg_2\|_0 + \|f_1 - g_1\|_0\|Dg_2\|_0.\end{aligned}$$

Proof. *i)* The C^0-estimate follows from the triangle inequality and submultiplicativity of the norm.

$$\begin{aligned}\|(f_1 \circ h)f_2 - (g_1 \circ h)g_2\|_0 &\leq \|(f_1 \circ h)(f_2 - g_2)\|_0 + \|(f_1 \circ h - g_1 \circ h)g_2\|_0 \\ &\leq \|f_1\|_0\|f_2 - g_2\|_0 + \|g_2\|_0\|f_2 - g_2\|_0.\end{aligned}$$

ii) For the C^1-estimate we use the product rule and triangle inequality to find

$$\begin{aligned}\|D[(f_1 \circ h)f_2] - D[(g_1 \circ h)g_2]\|_0 &= \|DF_1f_2 + F_1Df_2 - DG_1g_2 - G_1Dg_2\|_0 \\ &\leq \|DF_1f_2 - DG_1g_2\|_0 + \|F_1Df_2 - G_1Dg_2\|_0,\end{aligned}$$

where we introduce $F_1 = f_1 \circ h$ and $G_1 = g_1 \circ h$. We then estimate

$$\begin{aligned}\|DF_1f_2 - DG_1g_2\|_0 &\le \|DF_1f_2 - DF_1g_2\|_0 + \|DF_1g_2 - DG_1g_2\|_0 \\ &\le \|Df_1\|_0\|Dh\|_0\|f_2-g_2\|_0 + \|Df_1 - Dg_1\|_0\|Dh\|_0\|g_2\|_0, \\ \|F_1Df_2 - G_1Dg_2\|_0 &\le \|F_1Df_2 - F_1Dg_2\|_0 + \|F_1Dg_2 - G_2Dg_2\|_0 \\ &\le \|f_1\|_0\|Df_1 - Dg_2\|_0 + \|f_1-g_1\|_0\|Dg_2\|_0.\end{aligned}$$

For those estimates we have used that $DF_1 = Df_1(h)Dh$, and thus DF_1 is bounded by $\|Df_1\|_0\|Dh\|_0$ and likewise we have bounded $DF_1 - DG_1$ by $\|Df_1 - Dg_2\|_0\|Dh\|_0$. For the last estimate, we have used that F_1 is bounded by $\|f_1\|_0$ and $F_1 - G_1$ is bounded by $\|f_1 - g_1\|_0$. We obtain the desired estimate by adding the two estimates together □

Lemma 4.4. *Let $\rho, \tilde\rho \in C_b^1(X_c, \mathcal{L}(X_c, X_c))$ be such that $\|\rho\|_0, \|\tilde\rho\|_0 \le L_r$ and $\|D\rho\|_0, \|D\tilde\rho\|_0 \le \delta(\varepsilon)$ for some $\varepsilon > 0$. Furthermore, let X and Y be Banach spaces. Let $h \in C_b^2(Y, \mathcal{L}(Y, X))$ and $f_1, f_2 \in C_b^2(X_c, \mathcal{L}(X_c, Y))$. Furthermore, assume that $L_r < \|A_c^{-1}\|_{\text{op}}^{-1}$ and recall the definition of Q_ρ in (4.1.2).*

i) We have the C^0-estimate

$$\|h(f_1\circ T)Q_\rho - h(f_2\circ T)Q_{\tilde\rho}\|_0 \le \|h\|_0\|f_1\|_0L_{-1}^2\|\rho-\tilde\rho\|_0 + L_{-1}\|h\|_0\|f_1-f_2\|_0.$$

ii) We have the C^1-estimate

$$\begin{aligned}&\|D[h(f_1\circ T)Q_\rho] - D[h(f_2\circ T)Q_{\tilde\rho}]\|_0 \\ &\quad\le \|Dh\|_0\|f_1\|_0L_{-1}^2\|\rho-\tilde\rho\|_0 + L_{-1}\|Dh\|_0\|f_1-f_2\|_0 \\ &\qquad + \|h\|_0\|Df_1\|_0L_{-1}^3\|\rho-\tilde\rho\|_0 + L_{-1}^2\|h\|_0\|Df_1-Df_2\|_0 \\ &\qquad + 2\|h\|_0\|f_1\|_0L_{-1}^4\delta(\varepsilon)\|\rho-\tilde\rho\|_0 + \|h\|_0\|f_1\|_0L_{-1}^3\|D\rho - D\tilde\rho\|_0 \\ &\qquad + L_{-1}^3\delta(\varepsilon)\|h\|_0\|f_1-f_2\|_0.\end{aligned}$$

Proof. *i)* For the C^0-estimate, we note that $h(y)$ is a linear operator for all $y \in Y$ and we use submultiplicativity of the norm and Lemma 4.3i)

$$\|h(f_1\circ T)Q_\rho - h(f_2\circ T)Q_{\tilde\rho}\|_0 \le \|h\|_0\left(\|f_1\|_0\|Q_\rho - Q_{\tilde\rho}\|_0 + \|f_1-f_2\|_0\|Q_{\tilde\rho}\|_0\right).$$

All that is left to do is to show that $Q_\rho - Q_{\tilde\rho}$ is bounded by $L_{-1}^2\|\rho-\tilde\rho\|_0$ and that $Q_{\tilde\rho}$ is bounded L_{-1}. For the latter bound, we use similar calculations as performed at the end of the proof of Lemma 3.2i). Namely, fix $x \in X_c$, denote $y = T(x)$ and $\tau = Q_{\tilde\rho}(x) - A_c^{-1}$, then we have $(A_c^{-1} + \tau)(A_c + \tilde\rho(y)) = Q_{\tilde\rho}(x)P_{\tilde\rho}(y) = \text{Id}$. We can rewrite this as $\tau = -A_c^{-1}\tilde\rho(y)A_c^{-1} - \tau\tilde\rho(y)A_c^{-1}$. This implies that the norm of τ is bounded by $\|A_c^{-1}\|_{\text{op}}^2\|\tilde\rho(y)\|_{\text{op}}/\left(1 - \|A_c^{-1}\|_{\text{op}}\|\rho\tilde{(y)}\|_{\text{op}}\right) \le L_t$, as $\|\tilde\rho(y)\|_{\text{op}} \le L_r$, where L_r and L_t are defined in (2.0.2a) and (2.0.2b) respectively. Therefore, we have the desired bound

$$\|Q_{\tilde\rho}\|_0 \le \sup_{x\in X_c}\|Q_{\tilde\rho}(T(x))\|_{\text{op}} = \sup_{x\in X_c}\|A_c^{-1} + \tau(x)\|_{\text{op}} \le \sup_{x\in X_c}\|A_c^{-1}\|_{\text{op}} + L_t = L_{-1}.$$

The bound on $Q_\rho - Q_{\tilde\rho}$ now follows from submultiplicativity and

$$Q_\rho - Q_{\tilde\rho} = Q_\rho \left(P_{\tilde\rho}\circ T - P_\rho \circ T\right) Q_{\tilde\rho} = Q_\rho\left(\tilde\rho\circ T - \rho\circ T\right) Q_{\tilde\rho}.$$

ii) For the C^1-estimate, we start by with the product rule and triangle inequality to find

$$\begin{aligned}\|D[h(f_1\circ T)Q_\rho] - D[h(f_2\circ T)Q_{\tilde\rho}]\|_0 &\\ \le \|Dh\left(\mathrm{Id}, (f_1\circ T)(Q_\rho - Q_{\tilde\rho})\right)\|_0 + \|Dh\left(\mathrm{Id}, (f_1\circ T - f_2\circ T)Q_{\tilde\rho}\right)\|_0 &\\ + \|h\|_0\|D[(f_1\circ T)Q_\rho] - D[f_2\circ T)Q_{\tilde\rho}]\|_0. &\end{aligned}$$

The first two terms of the right hand side are estimated using similar arguments as those used in part i), that is

$$\begin{aligned}\|Dh\left(\mathrm{Id}, (f_1\circ T)(Q_\rho - Q_{\tilde\rho})\right)\|_0 &\le \|Dh\|_0\|f_1\|_0 L_{-1}^2\|\rho-\tilde\rho\|_0,\\ \|Dh\left(\mathrm{Id}, (f_1\circ T - f_2\circ T)Q_{\tilde\rho}\right)\|_0 &\le L_{-1}\|Dh\|_0\|f_1 - f_2\|_0,\end{aligned}$$

which are precisely the first two terms of the right hand side of our desired C^1-estimate. The last term can be estimated using Lemma 4.3ii):

$$\begin{aligned}\|D[(f_1\circ T)Q_\rho]-D[f_2\circ T)Q_{\tilde\rho}]\|_0 &\\ \le \|Df_1\|_0\|DT\|_0\|Q_\rho - Q_{\tilde\rho}\|_0 + \|Df_1 - Df_2\|_0\|DT\|_0\|Q_{\tilde\rho}\|_0 &\\ + \|f_1\|_0\|DQ_\rho - DQ_{\tilde\rho}\|_0 + \|f_1 - f_2\|_0\|DQ_{\tilde\rho}\|_0. &\end{aligned} \tag{4.2.1}$$

We will estimate the four terms separately. With the estimates of $Q_{\tilde\rho}$ and $Q_\rho - Q_{\tilde\rho}$ from the proof of part i), and given that $\|DT\|_0 \le L_{-1}$, we find

$$\begin{aligned}\|Df_1\|_0\|DT\|_0\|Q_\rho - Q_{\tilde\rho}\|_0 &\le \|Df_1\|_0 L_{-1}^3\|\rho-\tilde\rho\|_0,\\ \|Df_1 - Df_2\|_0\|DT\|_0\|Q_{\tilde\rho}\|_0 &\le L_{-1}^2\|Df_1 - Df_2\|_0,\end{aligned}$$

which are, up to the factor $\|h\|_0$, the third and fourth term of the right hand side of our desired C^1-estimate. Finally, we have to find an upper bound for $DQ_{\tilde\rho}$ and $DQ_\rho - DQ_{\tilde\rho}$ in terms of L_{-1} and $\|\rho-\tilde\rho\|_0$ to estimate the final two terms in (4.2.1). The product rule gives us, since $Q_{\tilde\rho} = \left(P_{\tilde\rho}\circ T\right)^{-1}$,

$$0 = D[Q_{\tilde\rho}(P_{\tilde\rho}\circ T)] = DQ_{\tilde\rho}(P_{\tilde\rho}\circ T) + Q_{\tilde\rho}DP_{\tilde\rho}(T)DT.$$

We isolate $DQ_{\tilde\rho}(P_{\tilde\rho}\circ T)$ and multiply from the right with $(P_{\tilde\rho}\circ T)^{-1} = Q_{\tilde\rho}$:

$$DQ_{\tilde\rho} = -Q_{\tilde\rho}DP_{\tilde\rho}(T)\left(DT, Q_{\tilde\rho}\right). \tag{4.2.2}$$

Furthermore, we note that $P_{\tilde\rho} = A_c + \tilde\rho$, hence $DP_{\tilde\rho} = D\tilde\rho$, which is bounded by $\delta(\varepsilon)$. We also saw that $\|Q_{\tilde\rho}\|_0 \le L_{-1}$ in the proof of part i). Hence we find with the triangle inequality

$$\begin{aligned}\|DQ_\rho - DQ_{\tilde\rho}\|_0 &\leq \|\left(Q_\rho - Q_{\tilde\rho}\right) DP_\rho(T)\left(DT, Q_\rho\right)\|_0 \\ &\quad + \|Q_{\tilde\rho}\left(DP_\rho(T) - DP_{\tilde\rho}(T)\right)\left(DT, Q_\rho\right)\|_0 \\ &\quad + \|Q_{\tilde\rho} DP_{\tilde\rho}(T)\left(DT, Q_\rho - Q_{\tilde\rho}\right)\|_0 \\ &\leq 2L_{-1}^4\delta(\varepsilon)\|\rho - \tilde\rho\|_0 + L_{-1}^3\|D\rho - D\tilde\rho\|_0.\end{aligned}$$

For the last inequality we used that $P_{\tilde\rho} = A_c + \tilde\rho$, and thus $DP_{\tilde\rho} = D\tilde\rho$, which is bounded by $\delta(\varepsilon)$. Furthermore, we used that $Q_{\tilde\rho}$ is bounded by L_{-1} and $Q_\rho - Q_{\tilde\rho}$ is bounded by $L_{-1}^2\|\rho - \tilde\rho\|_0$, as shown in the proof of part i). Hence the third factor of (4.2.1) is bounded by the fifth and sixth term appearing in the right hand side of our desired C^1-estimate. Finally, we estimate the last term of (4.2.1), where we use (4.2.2) to bound $DQ_{\tilde\rho}$:

$$\|f_1 - f_2\|_0\|DQ_{\tilde\rho}\|_0 \leq \|Q_{\tilde\rho}\|_0^2\|DT\|_0\|D\tilde\rho\|_0\|f_1 - f_2\|_0 \leq L_{-1}^3\delta(\varepsilon)\|f_1 - f_2\|_0.$$

This is precisely the final term appearing in the asserted estimate. □

4.3. *A new contraction*

With the previous two lemmas, we will show that $\Theta^{[2]}$ is a contraction on $\Gamma_2(\delta(\varepsilon))$ for $\varepsilon > 0$ small enough. We note again that we will later show that we can always scale our functions to satisfy the bound on the second derivative.

Proposition 4.5. *Assume that L_g and L_c are small in the sense of Remark 2.4 for $n = 2$. There exists an $\varepsilon_0 > 0$ such for all $\varepsilon < \varepsilon_0$ it holds that if $\|D^2 g\|_0, \|D^2 k_c\|_0 \leq \varepsilon$, then $\Theta^{[2]} : \Gamma_2(\delta(\varepsilon)) \to \Gamma_2(\delta(\varepsilon))$ is a contraction with respect to the C^1 norm.*

Proof. Let $\varepsilon > 0$ and $\|D^2 g\|_0, \|D^2 k_c\|_0 \leq \varepsilon$. Let $\mathcal{M} = \begin{pmatrix}\rho \\ \kappa_u \\ \kappa_s\end{pmatrix}, \tilde{\mathcal{M}} = \begin{pmatrix}\tilde\rho \\ \tilde\kappa_u \\ \tilde\kappa_s\end{pmatrix} \in \Gamma_2(\delta(\varepsilon))$. To show that $\Theta^{[2]}$ is a C^1 contraction, we will use the same steps as we used in the proof of Theorem 3.9.

A) We prove that $\Theta^{[2]}$ is a contraction with respect to the C^0 norm, independent of ε,
B) We show the existence of a constant $\theta_2(\varepsilon)$ such that

$$\|D[\Theta^{[2]}(\mathcal{M})] - D[\Theta^{[2]}(\tilde{\mathcal{M}})]\|_0 \leq \theta_2(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1,$$

C) We show that $\varepsilon > 0$ can be chosen so that $\theta_2(\varepsilon) < 1$, thus proving that $\Theta^{[2]}$ is a contraction with respect to the C^1 norm.

Step A) We recall that $\theta_1(\varepsilon)$, defined in (3.6.23), has the property $\theta_1(0) < 1$. We want to show that

$$\|\Theta^{[2]}(\mathcal{M}) - \Theta^{[2]}(\tilde{\mathcal{M}})\|_0 \leq \theta_1(0)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0.$$

Recall from equation (4.1.3) that

$$\Theta^{[2]}(\mathcal{M}) = \Theta^{[2]}\begin{pmatrix}\rho\\ \kappa_u\\ \kappa_s\end{pmatrix} = \begin{pmatrix} A_c Dk_c + Dg_c(K)\kappa - Dk_c\,(R)\,P_\rho \\ -A_u^{-1}Dg_u(K)\kappa + A_u^{-1}\kappa_u(R)P_\rho \\ A_s\kappa_s(T)Q_\rho + Dg_s(K\circ T)\kappa(T)Q_\rho \end{pmatrix},$$

which was derived by taking the derivative of $\Theta\begin{pmatrix} r\\ k_u\\ k_s\end{pmatrix}$. Therefore, we will use similar arguments as in step B) of the proof of Theorem 3.9 to show that

$$\|\Theta_i^{[2]}(\mathcal{M}) - \Theta_i^{[2]}(\tilde{\mathcal{M}})\|_0 \le \theta_{1,i}\|\mathcal{M}-\tilde{\mathcal{M}}\|_0$$

for $\theta_{1,i}$ given explicitly in equation (2.0.3a) to (2.0.3c) for $i = 1, 2, 3$.

ρ-component: We have

$$\begin{aligned}&\|\Theta_1^{[2]}(\mathcal{M}) - \Theta_1^{[2]}(\tilde{\mathcal{M}})\|_0 \\ &\quad\le \|(Dg_c\circ K)\kappa - (Dg_c\circ K)\tilde{\kappa}\|_0 + \|(Dk_c\circ R)P_\rho - (Dk_c\circ R)P_{\tilde{\rho}}\|_0.\end{aligned} \tag{4.3.1}$$

The first term is estimated by Lemma 4.3i):

$$\|(Dg_c\circ K)\kappa - (Dg_c\circ K)\tilde{\kappa}\|_0 \le \|Dg_c\|_0\|\kappa-\tilde{\kappa}\|_0 \le L_g\|\mathcal{M}-\tilde{\mathcal{M}}\|_0. \tag{4.3.2}$$

Here we recall that $\|Dg_c\|_0 \le L_g$, which follows from Assumption 4 of Theorem 2.1. Likewise, we estimate

$$\|(Dk_c\circ R)P_\rho - (Dk_c\circ R)P_{\tilde{\rho}}\|_0 \le \|Dk_c\|_0\|\rho-\tilde{\rho}\|_0 \le L_c\|\mathcal{M}-\tilde{\mathcal{M}}\|_0. \tag{4.3.3}$$

Thus inequality (4.3.1) together with estimates (4.3.2) and (4.3.3) gives

$$\|\Theta_1^{[2]}(\mathcal{M}) - \Theta_1^{[2]}(\tilde{\mathcal{M}})\|_0 \le \big(L_g + L_c\big)\|\mathcal{M}-\tilde{\mathcal{M}}\|_0 = \theta_{1,1}\|\mathcal{M}-\tilde{\mathcal{M}}\|_0. \tag{4.3.4}$$

κ_u-component: Similarly, we have

$$\begin{aligned}\|\Theta_2^{[2]}(\mathcal{M}) - \Theta_2^{[2]}(\tilde{\mathcal{M}})\|_0 &\le \|A_u^{-1}\|_{\text{op}}\|(Dg_u\circ K)\kappa - (Dg_u\circ K)\tilde{\kappa}\|_0 \\ &\quad + \|A_u^{-1}\|_{\text{op}}\|(\kappa_u\circ R)P_\rho - (\tilde{\kappa}_u\circ R)P_{\tilde{\rho}}\|_0.\end{aligned} \tag{4.3.5}$$

We again use Lemma 4.3i), which gives

$$\begin{aligned}\|(Dg_u\circ K)\kappa - (Dg_u\circ K)\tilde{\kappa}\|_0 &\le \|Dg_u\|_0\|\kappa-\tilde{\kappa}\|_0 \le L_g\|\mathcal{M}-\tilde{\mathcal{M}}\|_0,\\ \|(\kappa_u\circ R)P_\rho - (\tilde{\kappa}_u\circ R)P_{\tilde{\rho}}\|_0 &\le \|\kappa_u\|_0\|P_\rho - P_{\tilde{\rho}}\|_0 + \|P_{\tilde{\rho}}\|_0\|\kappa_u-\tilde{\kappa}_u\|_0 \\ &\le \big(L_u + \|A_c\|_{\text{op}} + L_r\big)\|\mathcal{M}-\tilde{\mathcal{M}}\|_0.\end{aligned}$$

Here we used that κ_u is bounded by L_u and $\tilde{\rho}$ by L_r. Thus inequality (4.3.5) becomes

$$\|\Theta_2^{[2]}(\mathcal{M}) - \Theta_2^{[2]}(\tilde{\mathcal{M}})\|_0 \le \theta_{1,2}\|\mathcal{M}-\tilde{\mathcal{M}}\|_0. \tag{4.3.6}$$

κ_s-component: Let us denote $G_s = Dg_s \circ K \circ T$. Then

$$\begin{aligned}\|\Theta_3^{[2]}(\mathcal{M}) - \Theta_3^{[2]}(\tilde{\mathcal{M}})\|_0 \leq \|&A_s(\kappa_s \circ T)Q_\rho - A_s(\tilde{\kappa}_s \circ T)Q_{\tilde{\rho}}\|_0 \\ &+ \|G_s(\kappa \circ T)Q_\rho - G_s(\tilde{\kappa} \circ T)Q_{\tilde{\rho}}\|_0. \end{aligned} \tag{4.3.7}$$

We will estimate both terms with Lemma 4.4i). We note that κ is bounded by $1 + L_c$, and hence we obtain

$$\|A_s(\kappa_s \circ T)Q_\rho - A_s(\tilde{\kappa}_s \circ T)Q_{\tilde{\rho}}\|_0 \leq \left(\|A_s\|_{\mathrm{op}} L_s L_{-1}^2 + L_{-1}\|A_s\|_{\mathrm{op}}\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0,$$

$$\|G_s(\kappa \circ T)Q_\rho - G_s(\tilde{\kappa} \circ T)Q_{\tilde{\rho}}\|_0 \leq \left(L_g(1 + L_c)L_{-1}^2 + L_{-1}L_g\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0.$$

Thus inequality (4.3.7) becomes

$$\|\Theta_3^{[2]}(\mathcal{M}) - \Theta_3^{[2]}(\tilde{\mathcal{M}})\|_0 \leq \theta_{1,3}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \tag{4.3.8}$$

Contraction constant: We can now estimate $\|\Theta^{[2]}(\mathcal{M}) - \Theta^{[2]}(\tilde{\mathcal{M}})\|_0$ with inequalities (4.3.4), (4.3.6) and (4.3.8). We have

$$\begin{aligned}\|\Theta^{[2]}(\mathcal{M}) - \Theta^{[2]}(\tilde{\mathcal{M}})\|_0 &= \max_{i=1,2,3}\left\{\|\Theta_i^{[2]}(\mathcal{M}) - \Theta_i^{[2]}(\tilde{\mathcal{M}})\|_0\right\} \\ &\leq \max_{i=1,2,3}\left\{\theta_{1,i}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0\right\} \\ &= \theta_1(0)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0, \end{aligned} \tag{4.3.9}$$

where the last equality follows from the definition of θ_1 in (3.6.23). Since $\theta_1(0) < 1$, we conclude that $\Theta^{[2]}$ is a contraction with respect to the C^0 norm.

Step B) Analogous to step B) of the proof of Theorem 3.9, we want to prove the componentwise inequality

$$\|D[\Theta_i^{[2]}(\mathcal{M})] - D[\Theta_i^{[2]}(\tilde{\mathcal{M}})]\|_0 \leq \left(\theta_{2,i} + C_{2,i}(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1,$$

with $\theta_{2,i}$ defined in (2.0.3a) to (2.0.3c) and $C_{2,i}$ will be defined during the proof. We note that $\mathcal{M}, \tilde{\mathcal{M}} \in \Gamma_2(\delta(\varepsilon))$, hence $D\rho$, $D\kappa_u$ and $D\kappa_s$ are bounded by $\delta(\varepsilon)$.

ρ-component: We start with

$$\begin{aligned}\|D[\Theta_1^{[2]}(\mathcal{M})] - D[\Theta_1^{[2]}(\tilde{\mathcal{M}})]\|_0 \leq \|&D[(Dg_c \circ K)\kappa] - D[(Dg_c \circ K)\tilde{\kappa}]\|_0 \\ &+ \|D[(Dk_c \circ R)P_\rho] - D[(Dk_c \circ R)P_{\tilde{\rho}}]\|_0. \end{aligned} \tag{4.3.10}$$

By applying Lemma 4.3ii) we find that

$$\begin{aligned}
&\|D[(Dg_c \circ K)\kappa] - D[(Dg_c \circ K)\tilde{\kappa}]\|_0 \\
&\qquad \le \|D^2 g_c\|_0 \|DK\|_0 \|\kappa - \tilde{\kappa}\|_0 + \|Dg_c\|_0 \|D\kappa - D\tilde{\kappa}\|_0 \\
&\qquad \le \big((1+L_c)\varepsilon + L_g\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1,
\end{aligned}$$

where we recall that DK is bounded by $1 + L_c$, see (3.3.3). Likewise, we find the estimate

$$\begin{aligned}
&\|D[(Dk_c \circ R)P_\rho] - D[(Dk_c \circ R)P_{\tilde{\rho}}]\|_0 \\
&\qquad \le \|D^2 k_c\|_0 \|DR\|_0 \|\rho - \tilde{\rho}\|_0 + \|Dk_c\|_0 \|D\rho - D\tilde{\rho}\|_0 \\
&\qquad \le \big(\big(\|A_c\|_{\mathrm{op}} + L_r\big)\, \varepsilon + 1 + L_c\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1.
\end{aligned}$$

Thus inequality (4.3.10) together with the above estimates gives

$$\begin{aligned}
\|D[\Theta_1^{[2]}(\mathcal{M})] - D[\Theta_1^{[2]}(\tilde{\mathcal{M}})]\|_0 &\le \big(L_g + L_c + \big(1 + L_c + \|A_c\|_{\mathrm{op}} + L_r\big)\, \varepsilon\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \\
&= \big(\theta_{2,1} + C_{2,1}(\varepsilon)\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1,
\end{aligned} \tag{4.3.11}$$

where we define $C_{2,1}(\varepsilon) := \big(1 + L_c + \|A_c\|_{\mathrm{op}} + L_r\big)\, \varepsilon$.

κ_u-component: We have

$$\begin{aligned}
\|D[\Theta_2^{[2]}(\mathcal{M})] - D[\Theta_2^{[2]}(\tilde{\mathcal{M}})]\|_0 &\le \|A_u^{-1}\|_{\mathrm{op}} \|D[(Dg_u \circ K)\kappa] - D[(Dg_u \circ K)\tilde{\kappa}]\|_0 \\
&\quad + \|A_u^{-1}\|_{\mathrm{op}} \|D[(\kappa_u \circ R)P_\rho] - D[(\tilde{\kappa}_u \circ R)P_{\tilde{\rho}}]\|_0.
\end{aligned} \tag{4.3.12}$$

By Lemma 4.3ii) we get

$$\begin{aligned}
\|D[(Dg_u \circ K)\kappa] - D[(Dg_u \circ K)\tilde{\kappa}]\|_0 &\le \big((1+L_c)\varepsilon + L_g\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1, \\
\|D[(\kappa_u \circ R)P_\rho] - D[(\tilde{\kappa}_u \circ R)P_{\tilde{\rho}}]\|_0 &\le \big((\|A_c\|_{\mathrm{op}} + L_r)\delta(\varepsilon) + L_u\big)\, \|\rho - \tilde{\rho}\|_1 \\
&\quad + \Big((\|A_c\|_{\mathrm{op}} + L_r)^2 + \delta(\varepsilon)\Big)\, \|\kappa_u - \tilde{\kappa}_u\|_1.
\end{aligned}$$

Thus inequality (4.3.12) becomes, as $\rho - \tilde{\rho}$ and $\kappa - \tilde{\kappa}$ are bounded by $\|\mathcal{M} - \tilde{\mathcal{M}}\|_1$,

$$\begin{aligned}
\|D[\Theta_2^{[2]}(\mathcal{M})] - D[\Theta_2^{[2]}(\tilde{\mathcal{M}})]\|_0 &\le \|A_u^{-1}\|_{\mathrm{op}} \Big((\|A_c\|_{\mathrm{op}} + L_r)^2 + L_g + L_u \\
&\qquad + (2 + L_c)\varepsilon + (\|A_c\|_{\mathrm{op}} + L_r)\delta\Big) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \\
&\le \big(\theta_{2,2} + C_{2,2}(\varepsilon)\big)\, \|\mathcal{M} - \tilde{\mathcal{M}}\|_1,
\end{aligned} \tag{4.3.13}$$

where we define $C_{2,2}(\varepsilon) := \|A_u^{-1}\|_{\mathrm{op}}((2 + L_c)\varepsilon + (\|A_c\|_{\mathrm{op}} + L_r)\delta(\varepsilon))$.

κ_s-component: Let us again denote $G_s = Dg_s \circ K \circ T$, then we have

$$\begin{aligned}\|D[\Theta_3^{[2]}(\mathcal{M})] - D[\Theta_3^{[2]}(\tilde{\mathcal{M}})]\|_0 \le \|D[A_s(\kappa_s \circ T)Q_\rho] - D[A_s(\tilde{\kappa}_s \circ T)Q_{\tilde{\rho}}]\|_0 \\ + \|D[G_s(\kappa \circ T)Q_\rho] - D[G_s(\tilde{\kappa} \circ T)Q_{\tilde{\rho}}]\|_0.\end{aligned} \tag{4.3.14}$$

We will estimate both terms with Lemma 4.4ii). For the first term, we note that A_s is the constant operator $x \mapsto A_s$, hence $\|A_s\|_0 = \|A_s\|_{\mathrm{op}}$ and $DA_s = 0$, which gives us

$$\begin{aligned}&\|D[A_s(\kappa_s \circ T)Q_\rho] - D[A_s(\tilde{\kappa}_s \circ T)Q_{\tilde{\rho}}]\|_0 \\ &\quad\le \|A_s\|_{\mathrm{op}} L_{-1}^3 \|D\kappa_s\|_0 \|\rho - \tilde{\rho}\|_0 + \|A_s\|_{\mathrm{op}} L_{-1}^2 \|D\kappa_s - D\tilde{\kappa}_s\|_0 \\ &\qquad + 2\|A_s\|_{\mathrm{op}} L_{-1}^4 \|\kappa_s\|_0 \delta(\varepsilon) \|\rho - \tilde{\rho}\|_0 + \|A_s\|_{\mathrm{op}} L_{-1}^3 \|\kappa_s\|_0 \|D\rho - D\tilde{\rho}\|_0 \\ &\qquad + \|A_s\|_{\mathrm{op}} L_{-1}^3 \delta(\varepsilon) \|\kappa_s - \tilde{\kappa}_s\|_0 \\ &\quad\le \|A_s\|_{\mathrm{op}} L_{-1}^2 \left(1 + L_{-1} L_s\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 &(4.3.15) \\ &\qquad + 2\|A_s\|_{\mathrm{op}} L_{-1}^3 \left(1 + L_{-1} L_s\right) \delta(\varepsilon) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1. &(4.3.16)\end{aligned}$$

The second term in the right hand side of (4.3.14) involves κ, which is bounded by $1 + L_c$, its derivative, which is bounded by

$$\|D\kappa\|_0 = \max\{D^2 k_c, D\kappa_u, D\kappa_s\} \le \max\{\varepsilon, \delta(\varepsilon)\} =: \gamma(\varepsilon),$$

and the derivative of G_s. The derivative of G_s is bounded by $\|D^2 g_s\|_0 \|DK\|_0 \|DT\|_0$, which in turn is bounded by $L_{-1}(1 + L_c)\varepsilon$. We obtain

$$\begin{aligned}&\|D[G_s(\kappa \circ T)Q_\rho] - D[G_s(\tilde{\kappa} \circ T)Q_{\tilde{\rho}}]\|_0 \\ &\quad\le \|DG_s\|_0 L_{-1}^2 \|\kappa\|_0 \|\rho - \tilde{\rho}\|_0 + \|DG_s\|_0 L_{-1} \|\kappa - \tilde{\kappa}\|_0 \\ &\qquad + \|G_s\|_0 L_{-1}^3 \|D\kappa\|_0 \|\rho - \tilde{\rho}\|_0 + \|G_s\|_0 L_{-1}^2 \|D\kappa - D\tilde{\kappa}\|_0 \\ &\qquad + 2\|G_s\|_0 L_{-1}^4 \|\kappa\|_0 \delta(\varepsilon) \|\rho - \tilde{\rho}\|_0 + \|G_s\|_0 L_{-1}^3 \|\kappa\|_0 \|D\rho - D\tilde{\rho}\|_0 \\ &\qquad + \|G_s\|_0 L_{-1}^3 \delta(\varepsilon) \|\kappa - \tilde{\kappa}\|_0 \\ &\quad\le L_{-1}^2 (L_g + L_{-1} L_g (1 + L_c)) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 &(4.3.17) \\ &\qquad + \left(L_{-1}^3 L_g \gamma(\varepsilon) + L_{-1}^3 (L_g + 2L_{-1} L_g (1 + L_c)) \delta(\varepsilon)\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 &(4.3.18) \\ &\qquad + \left(L_{-1}^3 (1 + L_c)^2 \varepsilon + L_{-1}^2 (1 + L_c) \varepsilon\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1.\end{aligned}$$

We see that (4.3.15) and (4.3.17) together are $\theta_{2,3}\|\mathcal{M} - \tilde{\mathcal{M}}\|_1$. Likewise, we can estimate (4.3.16) and (4.3.18) together by $2\theta_{3,3}\gamma(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1$ as $\delta(\varepsilon) \le \gamma(\varepsilon)$. Then inequality (4.3.14) becomes

$$\|D[\Theta_3^{[2]}(\mathcal{M})] - D[\Theta_3^{[2]}(\tilde{\mathcal{M}})]\|_0 \le \left(\theta_{2,3} + C_{2,3}(\varepsilon)\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1, \tag{4.3.19}$$

where we define $C_{2,3}(\varepsilon) := 2\theta_{3,3}\gamma(\varepsilon) + L_{-1}^2(1+L_c)\varepsilon + L_{-1}^3(1+L_c)^2\varepsilon$.

Lipschitz constant: Inequalities (4.3.11), (4.3.13) and (4.3.19) imply

$$\begin{aligned}
\|D[\Theta^{[2]}(\mathcal{M})] - D[\Theta^{[2]}(\tilde{\mathcal{M}})]\|_0 &= \max_{i=1,2,3} \left\{ \|D[\Theta_i^{[2]}(\mathcal{M})] - D[\Theta_i^{[2]}(\tilde{\mathcal{M}})]\|_0 \right\} \\
&\le \max_{i=1,2,3} \left\{ (\theta_{2,i} + C_{2,i}(\varepsilon)) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \right\} \\
&\le \theta_2(\varepsilon) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \qquad (4.3.20)
\end{aligned}$$

Here we define $\theta_2(\varepsilon) := \max\left\{\theta_{2,1}, \theta_{2,2}, \theta_{2,3}\right\} + \max\{C_{2,1}(\varepsilon), C_{2,2}(\varepsilon), C_{2,3}(\varepsilon)\}$.

Step C) From Remark 2.4 it follows that

$$\max\left\{\theta_{2,1}, \theta_{2,2}, \theta_{2,3}\right\} < 1.$$

As $\delta(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$, see Proposition 3.4, we have

$$\lim_{\varepsilon\to 0} \max\{C_{2,1}(\varepsilon), C_{2,2}(\varepsilon), C_{2,3}(\varepsilon)\} = 0.$$

For the limit of ε to 0 of $\theta_2(\varepsilon)$ we find

$$\lim_{\varepsilon\to 0} \theta_2(\varepsilon) = \max\left\{\theta_{2,1}, \theta_{2,2}, \theta_{2,3}\right\} < 1.$$

Therefore, we can find an $\varepsilon_0 > 0$ such that $\theta_2(\varepsilon) < 1$ for all $\varepsilon < \varepsilon_0$. Then estimates (4.3.9) and (4.3.20) give

$$\begin{aligned}
\|\Theta^{[2]}(\mathcal{M}) - \Theta^{[2]}(\tilde{\mathcal{M}})\|_1 &= \max\left\{ \|\Theta^{[2]}(\mathcal{M}) - \Theta^{[2]}(\tilde{\mathcal{M}})\|_0, \|D[\Theta^{[2]}(\mathcal{M})] - D[\Theta^{[2]}(\tilde{\mathcal{M}})]\|_0 \right\} \\
&\le \max\left\{ \theta_1(0)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0, \theta_2(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \right\} \\
&\le \lambda_2 \|\mathcal{M} - \tilde{\mathcal{M}}\|_1.
\end{aligned}$$

We define the contraction constant $\lambda_2 := \max\{\theta_1(0), \theta_2(\varepsilon)\}$, which is smaller than 1 by the above discussion. Thus we see that $\Theta^{[2]} : \Gamma_2(\delta(\varepsilon)) \to \Gamma_2(\delta(\varepsilon))$ is a contraction with respect to the C^1 norm for all $\varepsilon < \varepsilon_0$. $\square$

Corollary 4.6. *Let $\varepsilon > 0$ such that $F : X \to X$ satisfies the conditions of Theorems 2.1 and 3.9 and Proposition 4.5. Then the image of K is a C^2 center manifold for F.*

Proof. By assumption, $\varepsilon > 0$ is such that both Θ and $\Theta^{[2]}$ are contractions. Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$ be a fixed point of Θ. Then Corollary 3.10 implies that $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$ parameterizes a C^1 center manifold for F. Thus if we prove that Λ is C^2, or, equivalent, that $D\Lambda$ is C^1, we are done.

We will show that $D\Lambda$ is in fact the fixed point of $\Theta^{[2]}$. Let

$$\Gamma_{3/2} = \left\{ \mathcal{M} = \begin{pmatrix} \rho \\ \kappa_u \\ \kappa_s \end{pmatrix} \in C_b^0\left(X_c, \mathcal{L}(X_c, X)\right) \;\middle|\; \begin{array}{l} \mathcal{M}(0) = 0, \\ \|\rho\|_0 \le L_r \\ \|\kappa_u\|_0 \le \min\{L_u\} \\ \|\kappa_s\|_0 \le \min\{L_s\} \end{array} \right\},$$

then $D\Lambda \in \Gamma_{3/2}$ and $\Gamma_2(\delta(\varepsilon)) \subset \Gamma_{3/2}$. Furthermore, from step A) of the proof of the previous proposition, it follows that $\Theta^{[2]} : \Gamma_{3/2} \to \Gamma_{3/2}$ is a contraction with respect to the C^0 norm. Therefore, $D\Lambda$ is the unique fixed point of $\Theta^{[2]}$ in $\Gamma_{3/2}$. However, $\Theta^{[2]} : \Gamma_2(\delta(\varepsilon)) \to \Gamma_2(\delta(\varepsilon))$ is also a contraction with respect to the C^1 norm for ε sufficiently small. Let $\mathcal{M} \in \Gamma_2(\delta(\varepsilon))$ be the fixed point of $\Theta^{[2]}$. Then $\mathcal{M} \in \Gamma_{3/2}$ is a fixed point of $\Theta^{[2]}$, which means that $D\Lambda = \mathcal{M} \in \Gamma_2(\delta(\varepsilon))$. We conclude that $D\Lambda$ is C^1. $\square$

We have now shown in two steps that there exists a C^2 center manifold. In particular, we used the existence of a C^1 center manifold to obtain a C^2 center manifold in the second step. Furthermore, to obtain the C^1 center manifold, we explicitly used that our dynamical system F is at least C^2. Hence, with the current proof we cannot obtain a center manifold for a C^1 dynamical system.

We want to remark that if F is $C^{0+\mathrm{Lip}}$, we could slightly alter the definition of Γ_0 and show that Θ is a contraction with respect to the C^0 norm. Then both K and r would have been $C^{0+\mathrm{Lip}}$ and we would obtain a $C^{0+\mathrm{Lip}}$ center manifold and dynamical system.

Furthermore, if F is C^1 instead of C^2, we could adapt our proof to obtain a C^1 center manifold and dynamical system if X is uniformly convex, e.g. $X = \mathbb{R}^m$. In this case, we would also prove that Θ is a contraction with respect to the C^0 norm, which would give us $C^{0+\mathrm{Lip}}$ functions K and r. Using the results from [9], we know that both K and r would be almost everywhere differentiable. We could still define the fixed point operator $\Theta^{[2]}$ and prove that $\Theta^{[2]}$ would be a contraction with respect to the C^0 norm. Using similar arguments as in the previous corollary, we could then show that K and r would be everywhere continuously differentiable. Therefore, we would obtain a C^1 center manifold with C^1 dynamical system if we start with a C^1 dynamical system $F : X \to X$ on a uniformly convex space X.

5. A C^m center manifold

The final step of our proof scheme in Section 2.1 is inductively showing that the conjugacy is C^n. Similar to what we did in the C^2 case in the previous section, we will show that if the conjugacy is C^m, then the mth derivative will be C^1. We start again by defining a fixed point operator for the mth derivative of

$$\Lambda := \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix}.$$

To make the definition of the fixed point operator more insightful, we will use several lemmas before we define it. We start by stating Faà di Bruno's formula for derivatives of compositions, see for instance [14]

Lemma 5.1 *(Faà di Bruno's Formula). Let X, Y and Z be Banach spaces, and $f_1 : Y \to Z$ and $f_2 : X \to Y$ C^m functions. Then the mth derivative of $f_1 \circ f_2$ is given by*

$$D^m[f_1 \circ f_2](x) = \sum_{i=1}^{m} \sum_{\pi \in P_m^i} D^i f_1(f_2(x)) \left(D^{\pi(1)} f_2(x), \ldots, D^{\pi(i)} f_2(x) \right),$$

where P_m^i is the set of ordered partitions of length i of the set $\{1, \ldots, m\}$.

Since we want to define a fixed point operator for the mth derivative of Λ using $\Lambda = \Theta(\Lambda)$, we want to isolate the mth derivatives from Faà di Bruno's formula, and apply Faà di Bruno's formula to the composition of three functions.

Lemma 5.2. *Let X, Y and Z be Banach spaces, and $f_1 : Y \to Z$ and $f_2 : X \to Y$ C^m functions.*

i) The mth derivative of $f_1 \circ f_2$ is given by

$$D^m[f_1 \circ f_2](x) = Df_1(f_2(x))D^m f_2(x) + D^m f_1(f_2(x)) \, (Df_2(x))^{\otimes m} + \mathcal{P}_m(f_1, f_2)(x)$$

where we use the shorthand notation $(Df_2(x))^{\otimes m} := \underbrace{(Df_2(x), \ldots, Df_2(x))}_{m \text{ times}}$ and $\mathcal{P}_m(f_1, f_2)(x) := \sum_{i=2}^{m-1} \sum_{\pi \in P_m^i} D^i f_1(f_2(x)) \left(D^{\pi(1)} f_2(x), \cdots, D^{\pi(i)} f_2(x) \right)$.

ii) Let $f_3 : X \to X$ be another C^m function, then we find

$$\begin{aligned} D^m[f_1 \circ f_2 \circ f_3](x) = {} & Df_1(z) \, (Df_2(y)) \left(D^m f_3(x) \right) + Df_1(z) \left(D^m f_2(y) \right) (Df_3(x))^{\otimes m} \\ & + D^m f_1(z) \, (Df_2(y))^{\otimes m} \, (Df_3(x))^{\otimes m} + \mathcal{P}_m(f_1, f_2)(y) \, (Df_3(x))^{\otimes m} \\ & + \mathcal{P}_m(f_1 \circ f_2, f_3)(x), \end{aligned}$$

where $z = f_2(f_3(x))$ and $y = f_3(x)$.

Proof. *i)* The equality follows from Faà di Bruno's formula and the fact that there is one ordered partition of m of length 1 and one of length m.

ii) The equality follows from applying Faà di Bruno's formula twice. We first apply it to the composition of $f_1 \circ f_2$ and f_3, which gives

$$\begin{aligned} D^m[f_1 \circ f_2 \circ f_3](x) = {} & D[f_1 \circ f_2](f_3(x))D^m f_3(x) + D^m[f_1 \circ f_2](f_3(x)) \, (Df_3(x))^{\otimes m} \\ & + \mathcal{P}_m(f_1 \circ f_2, f_3)(x). \end{aligned}$$

Then the first derivative of $f_1 \circ f_2$ is given by $Df_1(f_2)Df_2$, and we apply Faà di Bruno's formula a second time for the mth derivative of $f_1 \circ f_2$ to find the desired equality. □

The third component of $\Theta(\Lambda)$ contains the function $(A_c + r)^{-1}$. We want to express the mth derivative in terms of the mth derivative of $A_c + r$.

Lemma 5.3. *Let $R = A_c + r : X_c \to X_c$ be an invertible C^m function with inverse T. Then*

$$D^m T = -DT\, D^m r(T)\,(DT)^{\otimes m} - DT\,\mathcal{P}_m(R,T) \qquad \text{for } m \geq 2.$$

Proof. We apply Faà di Bruno's formula to $R \circ T$, and notice that its mth derivative is 0 as $R \circ T = \mathrm{Id}$. Thus we find

$$DR(T) D^m T = -D^m r(T)\,(DT)^{\otimes m} - \mathcal{P}_m(R,T).$$

The asserted identity follows by multiplying both sides by the inverse of $DR(T)$, which is DT. □

Remark 5.4. This lemma is another reason why we start the inductive step after we have proven C^2 smoothness, because the first derivative of Id does not vanish.

5.1. Another fixed point operator

We can now take the mth derivative of the fixed point identity $\Lambda = \Theta(\Lambda)$, where $\Theta(\Lambda)$ is defined in (3.1.2). We will do this component-wise. For the first component we find, by Lemma 5.2i),

$$\begin{aligned} D^m r &= A_c D^m k_c + Dg_c(K) D^m K + D^m g_c(K)\,(DK)^{\otimes m} + \mathcal{P}_m(g_c, K) \\ &\quad - Dk_c(R) D^m r - D^m k_c(R)\,(DR)^{\otimes m} - \mathcal{P}_m(k_c, R) \\ &= f_{m,1} + Dg_c(K) D^m K + Dk_c(R) D^m r, \end{aligned}$$

where

$$f_{m,1} := A_c D^m k_c + D^m g_c(K)\,(DK)^{\otimes m} + \mathcal{P}_m(g_c, K) - D^m k_c(R)\,(DR)^{\otimes m} - \mathcal{P}_m(k_c, R).$$

For the second component, we find with Lemma 5.2i)

$$\begin{aligned} D^m k_u &= A_u^{-1}\left(Dk_u(R) D^m r + D^m k_u(R)\,(DR)^{\otimes m} + \mathcal{P}_m(k_u, R)\right) \\ &\quad - A_u^{-1}\left(Dg_u(K) D^m K + D^m g_u(K)\,(DK)^{\otimes m} + \mathcal{P}_m(g_u, K)\right) \\ &= f_{m,2} + A_u^{-1}\left(Dk_u(R) D^m r + D^m k_u(R)(DR)^{\otimes m} - Dg_u(K) D^m K\right), \end{aligned}$$

where

$$f_{m,2} := A_u^{-1}\left(\mathcal{P}_m(k_u, R) - D^m g_u(K)\,(DK)^{\otimes m} - \mathcal{P}_m(g_u, K)\right).$$

Finally, for the third component we find from Lemma 5.2ii) and Lemma 5.3, with $T = (A_c + r)^{-1}$,

$$
\begin{aligned}
D^m k_s &= A_s \left(Dk_s(T) D^m T + D^m k_s(T)\,(DT)^{\otimes m} + \mathcal{P}_m(k_s, T)\right) \\
&\quad + Dg_s(K \circ T)\,(DK(T))\left(D^m T\right) + Dg_s(K \circ T)\left(D^m K(T)\right)(DT)^{\otimes m} \\
&\quad + D^m g_s(K \circ T)\,(DK(T))^{\otimes m}\,(DT)^{\otimes m} + \mathcal{P}_m(g_s, K)(T)\,(DT)^{\otimes m} \\
&\quad + \mathcal{P}_m(g_s \circ K, T) \\
&= A_s\Big(- Dk_s(T) DT\, D^m r(T)\,(DT)^{\otimes} - Dk_s(T) DT \mathcal{P}_m(R, T) \\
&\quad + D^m k_s(T)\,(DT)^{\otimes m} + \mathcal{P}_m(k_s, T)\Big) \\
&\quad - Dg_s(K \circ T)\,(DK(T))\, DT\, D^m r(T)\,(DT)^{\otimes} - Dg_s(K(T)) DT \mathcal{P}_m(R, T) \\
&\quad + Dg_s(K \circ T)\left(D^m K(T)\right)(DT)^{\otimes m} + D^m g_s(K \circ T)\,(DK(T))^{\otimes m}\,(DT)^{\otimes m} \\
&\quad + \mathcal{P}_m(g_s, K)(T)\,(DT)^{\otimes m} + \mathcal{P}_m(g_s \circ K, T) \\
&= f_{m,3} - A_s Dk_s(T) h_m(D^m r) + A_s D^m k_s(T)\,(DT)^{\otimes m} \\
&\quad - Dg_s(K \circ T) DK(T) h_m(D^m r) + Dg_s(K \circ T) D^m K(T)\,(DT)^{\otimes m},
\end{aligned}
$$

where we define

$$
\begin{aligned}
f_{m,3} := {} & A_s\Big(- Dk_s(T) DT \mathcal{P}_m(R, T) + \mathcal{P}_m(k_s, T)\Big) - Dg_s(K(T)) DT \mathcal{P}_m(R, T) \\
& + D^m g_s(K \circ T)\,(DK(T))^{\otimes m}\,(DT)^{\otimes m} + \mathcal{P}_m(g_s, K)(T)\,(DT)^{\otimes m} + \mathcal{P}_m(g_s \circ K, T)
\end{aligned}
$$

and

$$
h_m(\rho)(x) := DT(x)\rho(T(x))(DT(x))^{\otimes m}.
$$

Hence, we introduce the fixed point operator, where we use $\kappa = \begin{pmatrix} D^m k_c \\ \kappa_u \\ \kappa_s \end{pmatrix}$,

$$
\begin{aligned}
& \Theta^{[m+1]} : \Gamma_{m+1} := C^1_b(X_c, \mathcal{L}^m(X_c, X)) \to C^1(X_c, \mathcal{L}^m(X_c, X)), \\
& \begin{pmatrix} \rho \\ \kappa_u \\ \kappa_s \end{pmatrix} \mapsto \begin{pmatrix} f_{m,1} + Dg_c(K)\kappa - Dk_c(R)\rho \\ f_{m,2} + A_u^{-1}\,(\kappa_u \circ R)\,(DR)^{\otimes m} + A_u^{-1} Dk_u(R)\rho - A_u^{-1} Dg_u(K)\kappa \\ f_{m,3} - A_s Dk_s(T) h_m(\rho) + A_s\,(\kappa_s \circ T)\,(DT)^{\otimes m} \\ \quad - Dg_s(K \circ T) DK(T) h_m(\rho) + Dg_s(K \circ T)\,(\kappa \circ T)\,(DT)^{\otimes m} \end{pmatrix}.
\end{aligned}
\tag{5.1.1}
$$

We note that if $\Lambda \in C^m_b(X_c, X)$, then the functions $f_{m,1}$, $f_{m,2}$ and $f_{m,3}$ are bounded, since we assume that $g \in C^m_b(X, X)$. Hence, if $\Lambda \in C^m_b(X_c, X)$, then we have that $\Theta^{[m+1]} : \Gamma_{m+1} \to \Gamma_{m+1}$.

Proposition 5.5. *Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_0$ be a C^m fixed point of Θ for $2 \le m < n$. If $\Lambda \in C^m_b(X_c, X)$, then $D^m \Lambda$ is a fixed point of $\Theta^{[m+1]} : \Gamma_{m+1} \to \Gamma_{m+1}$.*

Proof. This follows from the above discussion. □

5.2. Another contraction

We note that we can apply the estimates of Lemma 4.3 to terms such as $Dg_c(K)\kappa - Dg_c(K)\tilde{\kappa}$. However, we do not have estimates for terms such as $(\kappa_u \circ R)(DR)^{\otimes m} - (\tilde{\kappa}_u \circ R)(DR)^{\otimes m}$ with $\kappa_u, \tilde{\kappa}_u \in C_b^1(X_c, \mathcal{L}^m(X_c, X_c))$. Therefore, we start with some preliminary results about the differences of products.

Lemma 5.6. *Let X, Y and Z be Banach spaces, $m \geq 2$ and $h \in C_b^1(X, Y)$.*

i) Let $f, g \in C_b^0(Y, \mathcal{L}^m(Y, Z))$. For $\tilde{h} \in C_b^0(X, \mathcal{L}(X, Y))$ we have the C^0-estimate

$$\|(f \circ h)\tilde{h}^{\otimes m} - (g \circ h)\tilde{h}^{\otimes m}\|_0 \leq \|\tilde{h}\|_0^m \|f - g\|_0.$$

ii) Let $f, g \in C_b^1(X, \mathcal{L}^m(Y, Z))$. For $\tilde{h} \in C_b^1(X, \mathcal{L}(X, Y))$ we have the C^1-estimate

$$\begin{aligned}\|D[(f \circ h)\tilde{h}^{\otimes m}] - D[(g \circ h)\tilde{h}^{\otimes m}]\|_0 \leq{} & \|Dh\|_0 \|\tilde{h}\|_0^m \|Df - Dg\|_0 \\ & + m\|\tilde{h}\|_0^{m-1} \|D\tilde{h}\|_0 \|f - g\|_0.\end{aligned}$$

Proof. *i)* Let $x \in X$ and recall that the norm of the k-linear operator $(f - g)(h(x))$ is given by

$$\|(f - g)(h(x)\|_{\mathrm{op}} = \sup_{\substack{\|x_i\| \leq 1 \\ 1 \leq i \leq m}} \|(f - g)(h(x))(x_1, \dots, x_m)\|,$$

from which the desired C^0-estimate follows.

iii) For the C^1-estimate we use the product rule and triangle inequality to find

$$\begin{aligned}\|D[(f \circ h)\tilde{h}^{\otimes m}] - D[(g \circ h)\tilde{h}^{\otimes m}]\|_0 ={} & \|Df(h)\left(Dh, \tilde{h}^{\otimes m}\right) - Dg(h)\left(Dh, \tilde{h}^{\otimes m}\right)\|_0 \\ & + \|(f \circ h)D[\tilde{h}^{\otimes m}] - (g \circ h)D[\tilde{h}^{\otimes m}]\|_0.\end{aligned}$$

We note that $D[\tilde{h}^{\otimes m}] = \sum_{i=0}^{m-1} \left(\tilde{h}^{\otimes i}, D\tilde{h}, \tilde{h}^{\otimes m-1-i}\right)$, hence we find

$$\begin{aligned}\|D[(f \circ h)\tilde{h}^{\otimes m}] - D[(g \circ h)\tilde{h}^{\otimes m}]\|_0 \leq{} & \|Dh\|_0 \|D\tilde{h}\|_0^m \|Df - Dg\|_0 \\ & + \sum_{i=1}^{m-1} \|\tilde{h}\|_0^i \|D\tilde{h}\|_0 \|\tilde{h}\|_0^{m-1-i} \|f - g\|_0,\end{aligned}$$

from which the desired estimate follows. □

Proposition 5.7. *Let $m \geq 2$ and assume that L_g and L_c are small in the sense of Remark 2.4 for $n = m$. There exists an $\varepsilon_0 > 0$ such for all $\varepsilon < \varepsilon_0$ it holds that if $\|D^2 g\|_0, \|D^2 k_c\|_0 \leq \varepsilon$ and the fixed point $\Lambda \in \Gamma_1(\delta(\varepsilon))$ of Θ lies in $C_b^m(X_c, X)$, then $\Theta^{[m+1]} : \Gamma_{m+1} \to \Gamma_{m+1}$ is a contraction with respect to the C^1 norm.*

Proof. Let $\varepsilon > 0$ and $\|D^2 g\|_0, \|D^2 k_c\|_0 \le \varepsilon$. Let $\mathcal{M} = \begin{pmatrix} \rho \\ \kappa_u \\ \kappa_s \end{pmatrix}, \tilde{\mathcal{M}} = \begin{pmatrix} \tilde{\rho} \\ \tilde{\kappa}_u \\ \tilde{\kappa}_s \end{pmatrix} \in \Gamma_{\tilde{m}}$, where we denote $\tilde{m} = m + 1$.

To show that $\Theta^{[\tilde{m}]}$ is a C^1 contraction, we use the same steps as we used in the proofs of Theorem 3.9 and Proposition 4.5.

A) We prove that $\Theta^{[\tilde{m}]}$ is a contraction with respect to the C^0 norm, independent of ε,
B) We show the existence of a constant $\theta_{\tilde{m}}(\varepsilon)$ such that

$$\|D[\Theta^{[\tilde{m}]}(\mathcal{M})] - D[\Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 \le \theta_{\tilde{m}}(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1,$$

C) We show that $\varepsilon > 0$ can be chosen so that $\theta_{\tilde{m}}(\varepsilon) < 1$, thus showing that $\Theta^{[\tilde{m}]}$ is a contraction with respect to the C^1 norm.

Step A) We want to find $\theta_m < 1$ such that

$$\|\Theta^{[\tilde{m}]}(\mathcal{M}) - \Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 \le \theta_m \|\mathcal{M} - \tilde{\mathcal{M}}\|_0.$$

As we did in the proofs of Theorem 3.9 and Proposition 4.5, we will find the contraction constant component-wise, i.e. we will show that

$$\|\Theta_i^{[\tilde{m}]}(\mathcal{M}) - \Theta_i^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 \le \theta_{m,i}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0$$

for $\theta_{m,i}$ given explicitly in equation (2.0.3a) to (2.0.3c) and $i = 1, 2, 3$. Furthermore, we note that we will directly apply Lemma 4.3ii) and Lemma 5.6ii) to obtain the estimates (5.2.1), (5.2.2) and (5.2.4)

ρ-component: We recall that $\|Dg_c\|_0 \le L_g$ and $\|Dk_c\|_0 \le L_c$, which follows from Assumption 4 of Theorem 2.1. We estimate

$$\|\Theta_1^{[\tilde{m}]}(\mathcal{M}) - \Theta_1^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 \le \left(L_g + L_c\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0 = \theta_{m,1}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \tag{5.2.1}$$

κ_u-component: Similarly, we have the estimates $\|Dg_u\|_0 \le L_g$, $\|(DR)\|_0 \le \|A_c\|_{\text{op}} + L_r$ and $\|k_u\|_0 \le L_u$, hence we find

$$\begin{aligned}\|\Theta_2^{[\tilde{m}]}(\mathcal{M}) - \Theta_2^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 &\le \|A_u^{-1}\|_{\text{op}}\left((\|A_c\|_{\text{op}} + L_r)^m + L_u + L_g\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0 \\ &= \theta_{m,2}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \end{aligned}\tag{5.2.2}$$

κ_s-component: Finally, we have the bounds $\|DT\|_0 \le L_{-1}$, $\|Dk_s\|_0 \le L_s$, $\|Dg_s\|_0 \le L_g$, $\|DK\|_0 \le 1 + L_c$ and

$$\|h_m(\rho) - h_m(\tilde{\rho})\|_0 = \|DT\,(\rho \circ T - \tilde{\rho} \circ T)\,(DT)^{\otimes m}\|_0 \le L_{-1}^{m+1}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \tag{5.2.3}$$

We infer that

$$\begin{aligned}
&\|\Theta_3^{[\tilde{m}]}(\mathcal{M}) - \Theta_3^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 \\
&\qquad \le \left(\|A_s\|_{\text{op}}\left(L_s L_{-1}^{m+1} + L_{-1}^m\right) + L_g(1+L_c)L_{-1}^{m+1} + L_g L_{-1}^m\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_0 \\
&\qquad = \theta_{m,3}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \qquad (5.2.4)
\end{aligned}$$

<u>*Contraction constant:*</u> We can now estimate $\|\Theta^{[\tilde{m}]}(\mathcal{M}) - \Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0$ with inequalities (5.2.1), (5.2.2) and (5.2.4). We have

$$\begin{aligned}
\|\Theta^{[\tilde{m}]}(\mathcal{M}) - \Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0 &= \max_{i=1,2,3}\left\{\|\Theta_i^{[\tilde{m}]}(\mathcal{M}) - \Theta_i^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0\right\} \\
&\le \max_{i=1,2,3}\left\{\theta_{m,i}\|\mathcal{M} - \tilde{\mathcal{M}}\|_0\right\} \\
&= \theta_m\|\mathcal{M} - \tilde{\mathcal{M}}\|_0. \qquad (5.2.5)
\end{aligned}$$

Here we define $\theta_m := \max\left\{\theta_{m,1}, \theta_{m,2}, \theta_{m,3}\right\}$. We have assumed that Remark 2.4 holds for $n = m$. Therefore, we have $\theta_{m,i} < 1$ and thus $\theta_m < 1$. We conclude that $\Theta^{[\tilde{m}]}$ is a contraction with respect to the C^0 norm.

Step B) Analogous to step A), we want to prove the component-wise inequality

$$\|D[\Theta_i^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_i^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 \le \left(\theta_{\tilde{m},i} + C_{\tilde{m},i}(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1,$$

with $\theta_{\tilde{m},i}$ defined in (2.0.3a) to (2.0.3c) and $C_{\tilde{m},i}$ will be defined during the proof. We note that $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in \Gamma_1(\delta(\varepsilon))$, hence $\|D^2 r\|_0, \|D^2 k_u\|_0, \|D^2 k_s\|_0 \le \delta = \delta(\varepsilon)$.

<u>*ρ-component:*</u> Using the same estimates as we did for the ρ-component in step B) of the proof of Proposition 4.5, we find

$$\begin{aligned}
\|D[\Theta_1^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_1^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 &\le \left(L_g + (1+L_c)\varepsilon + L_c + (\|A_c\|_{\text{op}} + L_r)\varepsilon\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \\
&\le \left(\theta_{\tilde{m},1} + C_{\tilde{m},1}(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1, \qquad (5.2.6)
\end{aligned}$$

where we define $C_{\tilde{m},1}(\varepsilon) := \left(1 + L_c + \|A_c\|_{\text{op}} + L_r\right)\varepsilon$.

<u>*κ_u-component:*</u> We have

$$\begin{aligned}
&\|D[\Theta_2^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_2^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 \\
&\qquad \le \|A_u^{-1}\|_{\text{op}}\Big(\|D[(\kappa_u \circ R)(DR)^{\otimes m}] - D[(\tilde{\kappa}_u \circ R)(DR)^{\otimes m}]\|_0 \\
&\qquad\quad + \|D[Dk_u(R)\rho] - D[Dk_u(R)\tilde{\rho}]\|_0 \\
&\qquad\quad + \|D[Dg_u(K)\kappa] - D[Dg_u(K)\tilde{\kappa}]\|_0\Big). \qquad (5.2.7)
\end{aligned}$$

By applying Lemma 4.3ii) and Lemma 5.6ii) we find the estimates

$$\begin{aligned}
&\|D[(\kappa_u \circ R)(DR)^{\otimes m}] - D[(\tilde{\kappa}_u \circ R)(DR)^{\otimes m}]\|_0 \\
&\qquad\qquad\qquad \leq (\|A_c\|_{\mathrm{op}} + L_r)^{m+1} \|D\kappa_u - D\tilde{\kappa}_u\|_0 \\
&\qquad\qquad\qquad\quad + m(\|A_c\|_{\mathrm{op}} + L_r)^{m-1}\delta(\varepsilon)\|\kappa_u - \tilde{\kappa}_u\|_0, \\
&\|D[Dk_u(R)\rho] - D[Dk_u(R)\tilde{\rho}]\|_0 \leq \left((\|A_c\|_{\mathrm{op}} + L_r)\delta(\varepsilon) + L_u\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1, \\
&\|D[Dg_u(K)\kappa] - D[Dg_u(K)\tilde{\kappa}]\|_0 \leq \left((1 + L_c)\varepsilon + L_g\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1.
\end{aligned}$$

By combining these with (5.2.7) we estimate

$$\begin{aligned}
\|D[\Theta_2^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_2^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 &\leq \|A_u^{-1}\|_{\mathrm{op}} \Big((\|A_c\|_{\mathrm{op}} + L_r)^{\tilde{m}} + L_g + L_u \\
&\quad + \left(m(\|A_c\|_{\mathrm{op}} + L_r)^{m-1} + \|A_c\|_{\mathrm{op}} + L_r \right) \delta(\varepsilon) \\
&\quad + (1 + L_c)\varepsilon \Big) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \\
&\leq \left(\theta_{\tilde{m},2} + C_{\tilde{m},2}(\varepsilon)\right) \|\mathcal{M} - \tilde{\mathcal{M}}\|_1,
\end{aligned} \tag{5.2.8}$$

where we define

$$C_{\tilde{m},2}(\varepsilon) := \|A_u^{-1}\|_{\mathrm{op}}((1 + L_c)\varepsilon + (\|A_c\|_{\mathrm{op}} + L_r + m(\|A_c\|_{\mathrm{op}} + L_r)^{m-1}))\delta(\varepsilon).$$

κ_s-component: We have

$$\begin{aligned}
&\|D[\Theta_3^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_3^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 \\
&\qquad \leq \|A_s\|_{\mathrm{op}} \|D[Dk_s(T)h_m(\rho)] - D[Dk_s(T)h_m(\tilde{\rho})]\|_0 \\
&\qquad\quad + \|A_s\|_{\mathrm{op}} \|D[(\kappa_s \circ T)(DT)^{\otimes m}] - D[(\tilde{\kappa}_s \circ T)(DT)^{\otimes m}]\|_0 \\
&\qquad\quad + \|D[Dg_s(K \circ T)DK(T)h_m(\rho)] - D[Dg_s(K \circ T)DK(T)h_m(\tilde{\rho})]\|_0 \\
&\qquad\quad + \|D[Dg_s(K \circ T)(\kappa \circ T)(DT)^{\otimes m}] - D[Dg_s(K \circ T)(\tilde{\kappa} \circ T)(DT)^{\otimes m}]\|_0.
\end{aligned} \tag{5.2.9}$$

Before we apply Lemma 4.3ii), we start by deriving an upper bound for $Dh_m(\rho) - Dh_m(\tilde{\rho})$.

$$\begin{aligned}
Dh_m(\rho) &= D^2T\left(\mathrm{Id}, (\rho \circ T)(DT)^{\otimes m}\right) + DT\left(D\rho(T)\right)\left(DT, (DT)^{\otimes m}\right) \\
&\quad + DT(\rho \circ T) \sum_{i=0}^{m-1} \left((DT)^{\otimes i}, D^2T, (DT)^{\otimes m-1-i} \right).
\end{aligned}$$

From Lemma 5.3 we see that D^2T is bounded by $\|DT\, D^2r(T)\, (DT)^{\otimes 2}\|_0$ since $\mathcal{P}_2 = 0$. Using the estimates $\|DT\|_0 \leq L_{-1}$ and $\|D^2T\|_0 \leq L_{-1}^3\delta(\varepsilon)$, see (3.4.10) for the latter bound, we find

$$\|Dh_m(\rho) - Dh_m(\tilde{\rho})\|_0 \leq \left(L_{-1}^{m+3}\delta(\varepsilon) + L_{-1}^{m+2} + mL_{-1}^{m+3}\delta(\varepsilon) \right) \|\rho - \tilde{\rho}\|_1. \tag{5.2.10}$$

We can now apply Lemma 4.3ii) and estimate (5.2.3) for the bound on $h_m(\rho) - h_m(\tilde{\rho})$ to get

$$\begin{aligned}
&\|D[Dk_s(T)h_m(\rho)] - D[Dk_s(T)h_m(\tilde{\rho})]\|_0 \\
&\quad \le \|D^2 k_s\|_0 \|DT\|_0 \|h_m(\rho) - h_m(\tilde{\rho})\|_0 + \|Dk_s\|_0 \|Dh_m(\rho) - Dh_m(\tilde{\rho})\|_0 \\
&\quad \le \left(L_{-1}^{m+2}\delta(\varepsilon) + (m+1)L_{-1}^{m+3}L_s\delta(\varepsilon) + L_{-1}^{m+2}L_s\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \qquad (5.2.11)
\end{aligned}$$

By applying Lemma 5.6ii) we find

$$\begin{aligned}
&\|D[(\kappa_s \circ T)(DT)^{\otimes m}] - D[(\tilde{\kappa}_s \circ T)(DT)^{\otimes m}]\|_0 \\
&\quad \le \|DT\|_0 \|(DT)^{\otimes m}\|_0 \|D\kappa_s - D\tilde{\kappa}_s\|_0 + \|D(DT)^{\otimes m}\|_0 \|\kappa_s - \tilde{\kappa}_s\|_0 \\
&\quad \le \left(L_{-1}^{m+1} + mL_{-1}^{m+2}\delta(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \qquad (5.2.12)
\end{aligned}$$

For the third term of the right hand side of (5.2.9), we take $x \mapsto Dg_s(K(x))DK(x)$ for the functions f_1 and g_1 in Lemma 4.3ii) and use T as h. Recall that $\gamma(\varepsilon) = \max\{\varepsilon, \delta(\varepsilon)\}$, so that

$$\|D[Dg_s(K)DK]\|_0 \le \|D^2 g_s(DK, DK)\|_0 + \|Dg_s D^2 K\|_0 \le (1 + L_c)^2\varepsilon + L_g\gamma(\varepsilon).$$

Hence we find the estimate, using Lemma 4.3ii) and (5.2.3) and (5.2.10),

$$\begin{aligned}
&\|D[Dg_s(K \circ T)DK(T)h_m(\rho)] - D[Dg_s(K \circ T)DK(T)h_m(\tilde{\rho})]\|_0 \\
&\quad \le \|D[Dg_s(K)DK]\|_0 \|DT\|_0 \|h_m(\rho) - h_m(\tilde{\rho})\|_0 \\
&\qquad + \|Dg_s(K)DK\|_0 \|Dh_m(\rho) - Dh_m(\tilde{\rho})\|_0 \\
&\quad \le \left(L_{-1}^{m+2}L_g\gamma(\varepsilon) + (m+1)L_{-1}^{m+3}L_g(1 + L_c)\delta(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1 \qquad (5.2.13) \\
&\qquad + L_{-1}^{m+2}L_g(1 + L_c)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1 + L_{-1}^{m+2}(1 + L_c)^2\varepsilon\|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \qquad (5.2.14)
\end{aligned}$$

For the final term, we will apply Lemma 5.6, which gives

$$\begin{aligned}
&\|D[Dg_s(K \circ T)\kappa(T)(DT)^{\otimes m}] - D[Dg_s(K \circ T)\tilde{\kappa}(T)(DT)^{\otimes m}]\|_0 \\
&\quad \le \|D^2 g_s\|_0 \|D[K \circ T]\|_0 \|(\kappa \circ T)(DT)^{\otimes m} - (\tilde{\kappa} \circ T)(DT)^{\otimes m}\|_0 \\
&\qquad + \|Dg_s\|_0 \|D[(\kappa \circ T)(DT)^{\otimes m}] - D[(\tilde{\kappa} \circ T)(DT)^{\otimes m}]\|_0 \\
&\quad \le \|D^2 g_s\|_0 \|DK\|_0 \|DT\|_0 \|\kappa - \tilde{\kappa}\|_0 \|(DT)^{\otimes m}\|_0 \\
&\qquad + L_g\left(\|DT\|_0 \|(DT)^{\otimes m}\|_0 + \|D[(DT)^{\otimes m}]\|_0\right)\|\kappa - \tilde{\kappa}\|_1 \\
&\quad \le L_{-1}^{m+1}(1 + L_c)\varepsilon\|\mathcal{M} - \tilde{\mathcal{M}}\|_1 + \left(L_g L_{-1}^{m+1} + mL_g L_{-1}^{m+2}\delta(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \qquad (5.2.15)
\end{aligned}$$

We first recall that (5.2.11) and (5.2.12) should be multiplied by $\|A_s\|_{\mathrm{op}}$. We will collect the terms without ε in (5.2.11), (5.2.12) and (5.2.14), which sum up to $\theta_{\tilde{m},3}$. Then we collect the terms containing $\delta(\varepsilon)$ and $\gamma(\varepsilon)$ in (5.2.11), (5.2.12), (5.2.13) and (5.2.15). We estimate $\delta(\varepsilon)$ by

$\gamma(\varepsilon)$, so that those terms together are bounded by $\tilde{m}\theta_{\tilde{m}+1,3}\gamma(\varepsilon)$. Together with terms containing ε, we get

$$C_{\tilde{m},3}(\varepsilon) := \tilde{m}\theta_{\tilde{m}+1,3}\gamma(\varepsilon) + L_{-1}^{\tilde{m}}(1+L_c)\varepsilon + L_{-1}^{\tilde{m}+1}(1+L_c)^2\varepsilon.$$

Finally, we conclude that (5.2.9) reduces to

$$\|D[\Theta_3^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_3^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 \le \left(\theta_{\tilde{m},3} + C_{\tilde{m},3}(\varepsilon)\right)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1. \tag{5.2.16}$$

<u>*Lipschitz constant:*</u> Inequalities (5.2.6), (5.2.8) and (5.2.16) give

$$\begin{aligned}
\|D[\Theta^{[\tilde{m}]}(\mathcal{M})] - D[\Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0 &= \max_{i=1,2,3}\left\{\|D[\Theta_i^{[\tilde{m}]}(\mathcal{M})] - D[\Theta_i^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0\right\} \\
&\le \max_{i=1,2,3}\left\{(\theta_{\tilde{m},i} + C_{\tilde{m},i}(\varepsilon))\|\mathcal{M} - \tilde{\mathcal{M}}\|_1\right\} \\
&\le \theta_{\tilde{m}}(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1.
\end{aligned} \tag{5.2.17}$$

Here we define $\theta_{\tilde{m}}(\varepsilon) := \max\left\{\theta_{\tilde{m},1}, \theta_{\tilde{m},2}, \theta_{\tilde{m},3}\right\} + \max\{C_{\tilde{m},1}(\varepsilon), C_{\tilde{m},2}(\varepsilon), C_{\tilde{m},3}(\varepsilon)\}$.

Step C) From Remark 2.4 it follows that

$$\max\left\{\theta_{\tilde{m},1}, \theta_{\tilde{m},2}, \theta_{\tilde{m},3}\right\} < 1.$$

As $\delta(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$, see Proposition 3.4, we have

$$\lim_{\varepsilon\to 0}\max\{C_{\tilde{m},1}(\varepsilon), C_{\tilde{m},2}(\varepsilon), C_{\tilde{m},3}(\varepsilon)\} = 0.$$

Therefore, we can find an $\varepsilon_0 > 0$ such that $\theta_{\tilde{m}}(\varepsilon) < 1$ for all $\varepsilon < \varepsilon_0$. Then estimates (5.2.5) and (5.2.17) give

$$\begin{aligned}
&\|\Theta^{[\tilde{m}]}(\mathcal{M}) - \Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_1 \\
&\qquad = \max\left\{\|\Theta^{[\tilde{m}]}(\mathcal{M}) - \Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})\|_0, \|D[\Theta^{[\tilde{m}]}(\mathcal{M})] - D[\Theta^{[\tilde{m}]}(\tilde{\mathcal{M}})]\|_0\right\} \\
&\qquad \le \max\left\{\theta_m\|\mathcal{M} - \tilde{\mathcal{M}}\|_0, \theta_{\tilde{m}}(\varepsilon)\|\mathcal{M} - \tilde{\mathcal{M}}\|_1\right\} \\
&\qquad \le \lambda_{\tilde{m}}\|\mathcal{M} - \tilde{\mathcal{M}}\|_1.
\end{aligned}$$

Here we define the contraction constant $\lambda_{\tilde{m}} := \max\{\theta_m, \theta_{\tilde{m}}(\varepsilon)\}$ which is smaller than 1 by our previous discussion. We conclude that $\Theta^{[\tilde{m}]} : \Gamma_{\tilde{m}} \to \Gamma_{\tilde{m}}$ is a contraction with respect to the C^1 norm for all $\varepsilon < \varepsilon_0$. $\square$

Corollary 5.8. *Let $\varepsilon > 0$ be such that $F : X \to X$ satisfies the conditions of Theorems 2.1 and 3.9 and Propositions 4.5 and 5.7. Then the image of K is a C^n center manifold for F.*

Proof. The proof follows by induction on the smoothness of the conjugacy and the conjugate dynamics, with similar arguments as we had in the proof of Proposition 4.5. $\square$

6. Existence and uniqueness of the center manifold

6.1. *Proof of Theorem 2.1*

From Corollary 5.8 it follows that there exists an $\varepsilon > 0$ such that if $\|D^2 g\|_0$ and $\|D^2 k_c\|_0$ are smaller than ε, then there exists a C^n center manifold for $F : X \to X$. However, in Theorem 2.1, the only assumption on the second derivative of k_c and g is that they are both bounded. We will use a simple scaling argument to show that we can bound $D^2 k_c$ and $D^2 g$ by ε without affecting the assumed bound on Dk_c and Dg from Theorem 2.1.

Lemma 6.1. *Let $\varepsilon > 0$ and define $h^\varepsilon(x) := \varepsilon^{-1} h(\varepsilon x)$ for $h \in C_b^2(X, Y)$.*

i) We have $h^\varepsilon(0) = 0$ if $h(0) = 0$, $Dh^\varepsilon(0) = Dh(0)$ and $\|Dh^\varepsilon\|_0 = \|Dh\|_0$.
ii) We have $\|D^2 h^\varepsilon\|_0 = \varepsilon \|D^2 h\|_0$.
iii) For $h_1 \in C_b^2(Y, Z)$ and $h_2 \in C_b^2(X, Y)$ we have $h_1^\varepsilon \circ h_2^\varepsilon = (h_1 \circ h_2)^\varepsilon$.

Proof. These results follow from straightforward computations. □

We can now prove Theorem 2.1.

Proof of Theorem 2.1. Let $k_c : X_c \to X$ be chosen such that

$$k_c \in \left\{ h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \le L_c \right\}.$$

Then it follows from Lemma 6.1i) that for all $\varepsilon > 0$ we have

$$k_c^\varepsilon \in \left\{ h \in C_b^n(X_c, X_c) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \le L_c \right\}.$$

Likewise, if $F = A + g$ satisfies the conditions of Theorem 2.1, we have for all $\varepsilon > 0$ that $F^\varepsilon = A + g^\varepsilon$ with

$$g^\varepsilon \in \left\{ h \in C_b^n(X, X) \mid h(0) = 0,\ Dh(0) = 0 \text{ and } \|Dh\|_0 \le L_g \right\}.$$

Hence F^ε also satisfies the conditions of Theorem 2.1.

By Lemma 6.1ii) we can apply Corollary 5.8 to F^ε and k_c^ε for ε sufficiently small. We fix $\varepsilon > 0$ sufficiently small. Corollary 5.8 then provides a $K : X_c \to X$ which conjugates $f^\varepsilon : X \to X$ with $A_c + r : X_c \to X_c$. Then we find from Lemma 6.1iii) that

$$(F \circ K^{1/\varepsilon})^\varepsilon = F^\varepsilon \circ K = K \circ (A_c + r) = (K^{1/\varepsilon} \circ (A_c + r^{1/\varepsilon}))^\varepsilon,$$

and, again by Lemma 6.1iii),

$$F \circ K^{1/\varepsilon} = \left((F \circ K^{1/\varepsilon})^\varepsilon \right)^{1/\varepsilon} = \left(\left(K^{1/\varepsilon} \circ (A_c + r^{1/\varepsilon}) \right)^\varepsilon \right)^{1/\varepsilon} = K^{1/\varepsilon} \circ (A_c + r^{1/\varepsilon}).$$

Thus we see that $K^{1/\varepsilon} : X_c \to X$ conjugates $F : X \to X$ with $A_c + r^{1/\varepsilon} : X_c \to X_c$. Furthermore, by combining Lemma 6.1i) with Corollary 3.10, it follows that $r^{1/\varepsilon}$ satisfies Property A) and $k_u^{1/\varepsilon}$ and $k_s^{1/\varepsilon}$ satisfy Property B). □

6.2. Uniqueness of the center manifold

Since we found our conjugacy K and dynamical system $A_c + r$ with a contraction argument, we have uniqueness of $\begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix}$ in Γ_0. However, we can improve the uniqueness of k_u and k_s to arbitrary bounded functions, and the uniqueness of r to arbitrary bounded functions such that $A_c + r$ is invertible.

Lemma 6.2. *Let $F : X \to X$ and $k_c : X_c \to X_c$ satisfy the conditions of Theorem 2.1. Then*

$$F \circ \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} = \begin{pmatrix} \mathrm{Id} + k_c \\ k_u \\ k_s \end{pmatrix} \circ (A_c + r) \tag{6.2.1}$$

has a unique solution for $k_u \in C_b^0(X_c, X_u)$, $k_s \in C_b^0(X_c, X_s)$ and $r \in C_b^0(X_c, X_c)$ with the property that $A_c + r$ is a homeomorphism.

Proof. Let $\Lambda = \begin{pmatrix} r \\ k_u \\ k_s \end{pmatrix} \in C_b^n(X_c, X)$ be obtained from Theorem 2.1 and let $\mathcal{M} = \begin{pmatrix} \tilde{r} \\ \tilde{k}_u \\ \tilde{k}_s \end{pmatrix} \in C_b^0(X_c, X)$ be such that (6.2.1) holds. Then we know from Proposition 3.1 that $\mathcal{M} = \Theta(\mathcal{M})$ and $\Lambda = \Theta(\Lambda)$. We can mimic step A) of the proof of Theorem 3.9, since Lemma 3.5i) and Lemma 3.7i) can both be applied to the components of $\Theta(\mathcal{M}) - \Theta(\Lambda)$, even though $\mathcal{M}$ is merely C^0, see Remark 3.8. Hence we obtain

$$\|\mathcal{M} - \Lambda\|_0 = \|\Theta(\mathcal{M}) - \Theta(\Lambda)\|_0 \le \theta_0 \|\mathcal{M} - \Lambda\|_0.$$

Since $\theta_0 < 1$, we find $\mathcal{M} = \Lambda$. □

We conclude from Lemma 6.2 that, given $k_c : X_c \to X_c$, the conjugacy between the center dynamical system $R : X_c \to X_c$ and the original dynamical system $F : X \to X$ is unique in the space of continuous functions with bounded unstable and stable components. Additionally, we want to show that the center manifold is unique independent of our choice of $k_c : X_c \to X_c$. That is, we want to prove that the image of the conjugacy does not depend on a given $k_c : X_c \to X_c$.

Proposition 6.3. *Let $F : X \to X$ and $k_c, \tilde{k}_c : X_c \to X_c$ satisfy the conditions of Theorem 2.1. Then the image of $K = \iota + \begin{pmatrix} k_c \\ k_u \\ k_s \end{pmatrix}$ and $\tilde{K} = \iota + \begin{pmatrix} \tilde{k}_c \\ \tilde{k}_u \\ \tilde{k}_s \end{pmatrix}$ are the same, for $K, \tilde{K}$ the (unique) conjugacy obtained from Theorem 2.1.*

Proof. From Lemma 3.2i) it follows that $\mathrm{Id} + k_c$ is invertible, and from Lemma 3.2ii) we see that its inverse is given by $\mathrm{Id} + \psi$ for some bounded function $\psi : X_c \to X_c$. In particular, we can write

$$\mathrm{Id} + \tilde{k}_c = (\mathrm{Id} + k_c) \circ (\mathrm{Id} + \psi) \circ (\mathrm{Id} + \tilde{k}_c) = (\mathrm{Id} + k_c) \circ (\mathrm{Id} + \varphi),$$

where $\varphi = \psi \circ (\mathrm{Id} + \tilde{k}_c) + \tilde{k}_c$ is a bounded function. Likewise, $\mathrm{Id} + \tilde{k}_c$ is invertible, and thus $\mathrm{Id} + \varphi = (\mathrm{Id} + k_c)^{-1} \circ (\mathrm{Id} + \tilde{k}_c)$ is invertible, as it is the composition of two invertible functions. By Lemma 3.2i) its inverse is given by $\mathrm{Id} + \phi$ for some bounded function ϕ. We infer that

$$\begin{aligned} F \circ (K \circ (\mathrm{Id} + \varphi)) &= K \circ (A_c + r) \circ (\mathrm{Id} + \varphi) \\ &= (K \circ (\mathrm{Id} + \varphi)) \circ ((\mathrm{Id} + \phi) \circ (A_c + r) \circ (\mathrm{Id} + \varphi)) \\ &= \begin{pmatrix} \mathrm{Id} + \tilde{k}_c \\ k_u \circ (\mathrm{Id} + \varphi) \\ k_s \circ (\mathrm{Id} + \varphi) \end{pmatrix} \circ (A_c + \Phi), \end{aligned}$$

where we define $\Phi = r \circ (\mathrm{Id} + \varphi) + \phi \circ (A_c + r) \circ (\mathrm{Id} + \varphi)$. By Property A) of Theorem 2.1, we have that $A_c + r$ is invertible, and thus $A_c + \Phi$ is the composition of three invertible functions, hence invertible itself. Therefore, we use Lemma 6.2 to conclude $\tilde{k}_u = k_u \circ (\mathrm{Id} + \varphi)$ and $\tilde{k}_s = k_s \circ (\mathrm{Id} + \varphi)$. Since $\mathrm{Id} + \tilde{k}_c = (\mathrm{Id} + k_c) \circ (\mathrm{Id} + \varphi)$, we see that $\tilde{K} = K \circ (\mathrm{Id} + \varphi)$. As $\mathrm{Id} + \varphi$ is invertible, we conclude that

$$\mathrm{Im}(K) = K(X_c) = K((\mathrm{Id} + \varphi)(X_c)) = \tilde{K}(X_c) = \mathrm{Im}(\tilde{K}). \quad \square$$

6.3. *Proof of Corollary 2.3*

Finally, we want to show that if we have found approximations of the center manifold and the center dynamics, i.e., we have found an approximate conjugacy K_0 and approximate dynamical system R_0 such that

$$\| F \circ K_0 - K_0 \circ R_0 \|_m \le \varepsilon,$$

then the center manifold and dynamical system lie close to these approximations.

Proof of Corollary 2.3. Let $m < n$ and let $k_0 = \begin{pmatrix} k_{u,0} \\ k_{s,0} \end{pmatrix} : X_c \to X_u \oplus X_s$ and $r_0 : X_c \to X_c$. We use $\mathcal{M}$ to denote $\begin{pmatrix} r_0 \\ k_{u,0} \\ k_{s,0} \end{pmatrix}$ and $\mathcal{F} = F \circ \begin{pmatrix} k_c \\ k_{u,0} \\ k_{s,0} \end{pmatrix} - \begin{pmatrix} k_c \\ k_{u,0} \\ k_{s,0} \end{pmatrix} \circ (A_c + r_0)$. We assume that k_0 and r_0 satisfy the conditions of Corollary 2.3, that is, there exist constants $M > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} &k_0 \in \left\{ h \in C_b^{m+1}(X_c, X_u \oplus X_s) \,\middle|\, h(0) = 0,\ Dh(0) = 0 \text{ and } \|h\|_{m+1} \le M \right\}, \\ &r_0 \in \left\{ h \in C_b^{m+1}(X_c, X_c) \,\middle|\, h(0) = 0,\ Dh(0) = 0, \|Dh\|_0 \le L_r \text{ and } \|h\|_{m+1} \le M \right\}, \\ &\mathcal{F} \in \left\{ h \in C_b^m(X_c, X) \middle| \ \|h\|_m \le \varepsilon \right\}. \end{aligned}$$

Our proof consists of the following steps:

1. We prove that if $\mathcal{F}$ is small in C^m, then $\mathcal{M} - \Theta(\mathcal{M})$ is small in C^m.
2. We prove that if $\mathcal{M} - \Theta(\mathcal{M})$ is small in C^0, then $\mathcal{M} - \Lambda$ is small in C^0, where Λ is the fixed point of Θ.

3. Using induction, we prove that if $\mathcal{M} - \Theta(\mathcal{M})$ is small in C^m, then $D^m\mathcal{M} - D^m\Lambda$ is small in C^0.

For the first step, we want an explicit estimate for $\mathcal{M} - \Theta(\mathcal{M})$. By definition of Θ, see (3.1.3), we have

$$\mathcal{M} - \Theta(\mathcal{M}) = \begin{pmatrix} -\mathcal{F}_1 \\ A_u^{-1}\mathcal{F}_2 \\ -\mathcal{F}_3 \circ (A_c + r_0)^{-1} \end{pmatrix}.$$

We can clearly estimate the C^m norm of the first two components by $\|\mathcal{F}\|_m \leq \varepsilon$ and $\|A_u^{-1}\|_{\text{op}}\|\mathcal{F}\|_m \leq \|A_u^{-1}\|_{\text{op}}\varepsilon$ respectively. For the third component, we use Faà di Bruno's formula and Lemma 5.3 to obtain an estimate. We find

$$\begin{aligned}\|\mathcal{F}_3 \circ (A_c + r_0)^{-1}\|_m &\leq \mathcal{C}\left(D(A_c + r_0)^{-1}, D^i(A_c + r_0) \text{ for } i \leq m\right)\|\mathcal{F}_3\|_m \\ &\leq \mathcal{C}_1(M, m)\varepsilon,\end{aligned}$$

where we used that the derivatives of r_0 are bounded by M and $D(A_c + r_0)^{-1}$ is bounded by L_{-1} as Dr_0 is bounded by L_r. Hence we obtain

$$\|\mathcal{M} - \Theta(\mathcal{M})\|_m \leq \max\{1, \|A_u^{-1}\|_{\text{op}}, \mathcal{C}_1(M, m)\}\varepsilon = \mathcal{C}_2(M, m)\varepsilon. \tag{6.3.1}$$

Here $\mathcal{C}_2(M, m) := \max\{1, \|A_u^{-1}\|_{\text{op}}, \mathcal{C}_1(M, m)\}$. Hence we have shown that if $\mathcal{F}$ is small, then $\mathcal{M}$ is an almost fixed point of Θ.

To prove the second step, we use that Θ is a contraction in the C^0 norm with contraction constant θ_0, see the proof of Lemma 6.2, thus we have

$$\begin{aligned}\|\Lambda - \mathcal{M}\|_0 &\leq \|\Theta(\Lambda) - \Theta(\mathcal{M})\|_0 + \|\mathcal{M} - \Theta(\mathcal{M})\|_0 \\ &\leq \theta_0\|\Lambda - \mathcal{M}\|_0 + \|\mathcal{M} - \Theta(\mathcal{M})\|_0.\end{aligned}$$

By rewriting, we obtain the estimate

$$\|\Lambda - \mathcal{M}\|_0 \leq \frac{1}{1 - \theta_0}\|\mathcal{M} - \Theta(\mathcal{M})\|_0. \tag{6.3.2}$$

Combining (6.3.1) and (6.3.2) proves Corollary 2.3 for $m = 0$, that is

$$\|\Lambda - \mathcal{M}\|_0 \leq \frac{\mathcal{C}_2(M, 0)}{1 - \theta_0}\varepsilon =: \mathcal{C}(M, 0)\varepsilon.$$

To prove step 3, we will use induction. So let us assume that $\|\Lambda - \mathcal{M}\|_{m-1} \leq \mathcal{C}(M, m-1)\varepsilon$. By construction of the contraction $\Theta^{[m+1]}$, its fixed point is $D^m\Lambda$. Thus similarly to (6.3.2), we have

$$\|D^m\Lambda - D^m\mathcal{M}\|_0 \leq \frac{1}{1 - \theta_m}\|D^m\mathcal{M} - \Theta^{[m+1]}(D^m\mathcal{M})\|_0. \tag{6.3.3}$$

From (6.3.1), we know that $\| D^m (\mathcal{M} - \Theta(\mathcal{M})) \|_0 \leq \mathcal{C}_1(M, m)\varepsilon$. Hence we estimate (6.3.3) with the triangle inequality and the estimate on $D^m (\mathcal{M} - \Theta(\mathcal{M}))$ to obtain

$$\| D^m \Lambda - D^m \mathcal{M} \|_0 \leq \frac{\mathcal{C}_2(M, m)\varepsilon}{1 - \theta_m} + \frac{\| D^m \Theta(\mathcal{M}) - \Theta^{[m+1]}(D^m \mathcal{M}) \|_0}{1 - \theta_m}. \tag{6.3.4}$$

Therefore, it remains to find a bound on $D^m \Theta(\mathcal{M}) - \Theta^{[m+1]}(D^m \mathcal{M})$. Recall from (3.1.2) the definition of Θ. By Faà di Bruno's formula, there exists a function $\mathcal{G}$ such that for all $\mathcal{T} : X_c \to X$

$$D^m \Theta(\mathcal{T}) = \mathcal{G}(\mathcal{T}, D\mathcal{T}, \ldots, D^m \mathcal{T}). \tag{6.3.5}$$

By construction, $\mathcal{G}(\mathcal{T}, \ldots, D^m \mathcal{T})(x)$ is a linear combination of products of various derivatives of $\mathcal{T}$, evaluated at either x or $\mathcal{T}(x)$. In particular, we have for $\mathcal{M} : X_c \to X$

$$D^m \Theta(\mathcal{M}) = \mathcal{G}(\mathcal{M}, D\mathcal{M}, \ldots, D^{m-1} \mathcal{M}, D^m \mathcal{M}). \tag{6.3.6}$$

On the other hand, by definition of $\Theta^{[m+1]}$, see (5.1.1), we also have that

$$\Theta^{[m+1]}(D^m \mathcal{M}) = \mathcal{G}(\Lambda, D\Lambda, \ldots, D^{m-1} \Lambda, D^m \mathcal{M}). \tag{6.3.7}$$

Subtracting (6.3.7) from (6.3.6) we thus obtain

$$D^m \Theta(\mathcal{M}) - \Theta^{[m+1]}(D^m \mathcal{M}) = \int_0^1 D\mathcal{G}(\mathcal{N}(s), D^m \mathcal{M}) \mathrm{d}s \begin{pmatrix} \Lambda - \mathcal{M} \\ \vdots \\ D^{m-1} (\Lambda - \mathcal{M}) \\ 0 \end{pmatrix}, \tag{6.3.8}$$

where $\mathcal{N}(s) = (1 - s) \left(\mathcal{M}, \ldots, D^{m-1} \mathcal{M} \right) + s \left(\Lambda, \ldots, D^{m-1} \Lambda \right)$.

To calculate the partial derivative of $\mathcal{G}$ in the direction of its first input, we use the chain rule and obtain an expression depending on $\mathcal{G}$, $\mathcal{N}(s)$, $D\mathcal{N}(s)$, $D^m \mathcal{M}$ and $D^{m+1} \mathcal{M}$. The other partial derivatives of $\mathcal{G}$ are partial derivatives of polynomials, hence only depend on $\mathcal{G}$, $\mathcal{N}(s)$ and $D^m \mathcal{M}$. In particular, $D\mathcal{G}$ is continuous and evaluated on the compact set $\{(\mathcal{N}(s), D^m \mathcal{M}) \mid s \in [0, 1]\}$ in (6.3.8) – note that $\Lambda, \mathcal{M} \in C_b^{m+1}(X_c, X)$. Thus $\| D\mathcal{G}(\mathcal{N}(s), D^m \mathcal{M}) \|_{\mathrm{op}} \leq \mathcal{C}_3(M, m)$ for all $s \in [0, 1]$. Therefore, we can estimate (6.3.8) by

$$\begin{aligned} \| D^m \Theta(\mathcal{M}) - \Theta^{[m+1]}(D^m \mathcal{M}) \|_0 &\leq \mathcal{C}_3(M, m) \| \Lambda - \mathcal{M} \|_{m-1} \\ &\leq \mathcal{C}_3(M, m)\mathcal{C}(M, m - 1)\varepsilon. \end{aligned}$$

Using (6.3.4) and our induction hypothesis, we thus conclude that

$$\begin{aligned} \| \Lambda - \mathcal{M} \|_m &= \max \left\{ \| \Lambda - \mathcal{M} \|_{m-1}, \| D^m \Lambda - D^m \mathcal{M} \|_0 \right\} \\ &\leq \max \left\{ \mathcal{C}(M, m - 1), \frac{\mathcal{C}_2(M, m) + \mathcal{C}_3(M, m)\mathcal{C}(M, m - 1)}{1 - \theta_m} \right\} \varepsilon \\ &=: \mathcal{C}(M, m)\varepsilon. \quad \square \end{aligned}$$

6.4. Guide towards applications

For practical applications of Theorem 2.1 and Corollary 2.3, one has to make sure that the non-linearity of F is bounded and find approximate solutions k_0 and r_0 of the conjugacy equation (2.0.1). The approximation r_0 and the explicit error bound from Corollary 2.3 allows one to analyze the dynamical behavior on the center manifold. We will outline both steps as a stepping stone for future applications. We refer the interested reader to [17] for a detailed presentation of an application of the parameterization method for center manifolds.

As mentioned above, in most practical applications the dynamical system $F = A + g$ does not automatically satisfy the conditions of Theorem 2.1 because the non-linear part g is unbounded. One way to overcome this problem is by multiplying g with a smooth cut-off function ξ_g. The center manifold we obtain for $F_\xi = A + g \cdot \xi_g$ will be a local center manifold for F in the region where $\xi_g \equiv 1$. By shrinking the support of ξ_g, we can make $\|D[g\xi_g]\|_0 \leq \|(Dg) \cdot \xi_g\|_0 + \|g \cdot D\xi_g\|_0$ as small as desired.

To find approximations k_0 and r_0, one can use Taylor series. To obtain Taylor approximations P_k and P_r for $k_u \oplus k_s$ and r respectively, we can solve $F_\xi \circ K - K \circ R = 0$ recursively, i.e. order by order. Our choice of k_c influences the Taylor approximations P_k and P_r. Choosing k_c such that the Taylor approximation P_r is in normal form makes it easier to analyze the conjugate dynamics on the center manifold once we obtain the error bounds from Corollary 2.3.

However, the Taylor approximations P_k and P_r are polynomials, hence unbounded functions. Therefore, we cannot use P_k and P_r directly as approximate solutions k_0 and r_0. To find bounds on $P_k - k_u \oplus k_s$ and $P_r - r$, there are two strategies one can use:

For the first strategy, we again use cut-off functions to obtain bounded functions $k_0 = P_k\xi_k$ and $r_0 = P_r\xi_r$ which we can use as approximate solutions in (2.0.1). Corollary 2.3 converts a bound on the residue of the conjugacy equation into an explicit bound on the C^m norm of $k_0 - k_u \oplus k_s$ and $r_0 - r$. In particular, on the region where $\xi_g \circ K \equiv 1$, $\xi_k \equiv 1$ and $\xi_r \equiv 1$, the same explicit bound is a bound on $P_k - k_u \oplus k_s$ and $P_r - r$ for the local center manifold of F. The main challenge one faces is choosing the initial cut-off function ξ_g and the additional cut-off functions ξ_k and ξ_r such that $F_\xi \circ k_0 - k_0 \circ r_0$ is uniformly small away from the origin. Around the origin $F_\xi \circ k_0 - k_0 \circ r_0$ behaves as $\|x\|^{n+1}$ once we solved $F_\xi \circ K - K \circ R$ up to order n.

For the second strategy, we simply use $k_0 = 0$ and $r_0 = 0$ as approximate solutions. Corollary 2.3 then gives an explicit bound on the C^m norm of $k_u \oplus k_s$ and r. In particular this is a bound on the mth derivative of $k_u \oplus k_s$ and r. Hence, with Taylor's theorem we obtain an explicit bound on $P_k - k_u \oplus k_s$ and $P_r - r$. The main advantage of this strategy is that $F_\xi \circ k_0 - k_0 \circ r_0$ will be uniformly small (without the need for additional cut-off functions). However, the residue of $F_\xi \circ k_0 - k_0 \circ r_0$ will be larger than the residue in the previous method, as $F_\xi \circ k_0 - k_0 \circ r_0$ behaves quadratically near the origin.

References

[1] Inmaculada Baldomá, Ernest Fontich, Pau Martín, Invariant manifolds of parabolic fixed points (I). Existence and dependence on parameters, ArXiv e-prints (2016), available at https://arxiv.org/abs/1603.02533.

[2] Inmaculada Baldomá, Ernest Fontich, Pau Martín, Invariant manifolds of parabolic fixed points (II). Approximations by sums of homogeneous functions, ArXiv e-prints (2016), available at https://arxiv.org/abs/1603.02535.

[3] Jan Bouwe van den Berg, Christian Reinhardt, J.D. Mireles James, Computing (un)stable manifolds with validated error bounds: non-resonant and resonant spectra, J. Nonlinear Sci. 26 (4) (2015) 1055–1095.

[4] Xavier Cabré, Ernest Fontich, Rafael de la Llave, The parameterization method for invariant manifolds I: manifolds associated to non-resonant subspaces, Indiana Univ. Math. J. (2003) 283–328.

[5] Xavier Cabré, Ernest Fontich, Rafael de la Llave, The parameterization method for invariant manifolds II: regularity with respect to parameters, Indiana Univ. Math. J. (2003) 329–360.

[6] Xavier Cabré, Ernest Fontich, Rafael de la Llave, The parameterization method for invariant manifolds III: overview and applications, J. Differ. Equ. 218 (2) (2005) 444–515.

[7] Jack Carr, Applications of Centre Manifold Theory, Springer US, 1982.

[8] Roberto Castelli, Jean-Philippe Lessard, J.D. Mireles James, Parameterization of invariant manifolds for periodic orbits (II): a posteriori analysis and computer assisted error bounds, J. Dyn. Differ. Equ. 30 (4) (2018) 1525–1581.

[9] James A. Clarkson, Uniformly convex spaces, Trans. Am. Math. Soc. 40 (3) (1936) 396–414.

[10] Ariadna Farrés, Àngel Jorba, On the high order approximation of the centre manifold for odes, Discrete Contin. Dyn. Syst., Ser. B 14 (3) (2010) 977–1000.

[11] Chris Groothedde, J.D. Mireles James, Parameterization method for unstable manifolds of delay differential equations, J. Comput. Dyn. 4 (1&2) (2017) 21–70.

[12] Nicholas Karydas, John Schinas, The center manifold theorem for a discrete system, Appl. Anal. 44 (3–4) (1992) 267–284.

[13] Jean-Philippe Lessard, J.D. Mireles James, Christian Reinhardt, Computer assisted proof of transverse saddle-to-saddle connecting orbits for first order vector fields, J. Dyn. Differ. Equ. 26 (2) (2014) 267–313.

[14] Andrea Mennucci, An intuitive presentation of faà di bruno's formula, in: L'eredità Matematica e Civile di Francesco Faà di Bruno, 2017.

[15] Roy Plastock, Homeomorphisms between Banach spaces, Trans. Am. Math. Soc. 200 (1974) 169–183.

[16] Christian Reinhardt, J.D. Mireles James, Fourier–Taylor parameterization of unstable manifolds for parabolic partial differential equations: formalism, implementation and rigorous validation, Indag. Math. 30 (1) (2019) 39–80.

[17] Jan Bouwe van den Berg, Wouter Hetebrij, Bob Rink, Generalizations of the parameterization method for center manifolds, In preparation.

[18] Jan Bouwe van den Berg, J.D. Mireles James, Jean-Philippe Lessard, Konstantin Mischaikow, Rigorous numerics for symmetric connecting orbits: even homoclinics of the Gray-Scott equation, SIAM J. Math. Anal. 43 (4) (2011) 1557–1594.

[19] Andre Vanderbauwhede, Centre manifolds, normal forms and elementary bifurcations, in: Urs Kirchgraber, Hans Otto Kirchgraber (Eds.), Dynamics Reported: a Series in Dynamical Systems and Their Applications, Vieweg+Teubner Verlag, Wiesbaden, 1989, pp. 89–169.